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THE ELEMENTS

OF

PLANE GEOMETRY.

Association for the Improvement of Geometrical Teaching.

THE

ELEMENTS OF PLANE GEOMETRY

PART I.

(CORRESPONDING TO EUCLID BOOKS I.–II.).

Prepared by the Committee appointed by the Association.

"Proficiency in Pure Geometry . . . . . . perennially a symptom, not only of steady application, but of a clear methodic intellect, and offering in all epochs good promise for all manner of arts and pursuits."—T. CARLYLE.

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PREFACE.

THE Association for the Improvement of Geometrical Teaching was formed in the year 1871, and published in the year 1875 a "Syllabus of Plane Geometry (corresponding to Euclid Books I.–VI.)." This Syllabus has received a considerable amount of favourable recognition at the hands of Teachers and others interested in the study of Geometry, as is proved by the fact that an edition of 3000 copies has recently been exhausted and, to meet a steadily continuing demand, a new edition has been published. It was felt, however, by many members of the Association, that in order to secure for the Syllabus more general consideration and acceptance, it was desirable that an authorised series of Proofs of the Propositions contained in it should be issued, and accordingly at the Annual Meeting in January, 1881, it was resolved—

"That a sub-committee be appointed to draw up Proofs of the Propositions of the Syllabus of Plane Geometry."

The result of the labours of the committee appointed in accordance with this resolution, extending over Books I. and II. of the Syllabus, was approved at a General Meeting of the Association, held on the 20th March last, and is submitted to the public in the following pages.

The Association is desirous that it should be clearly understood that the present work is not offered as a section of a complete treatise on Elementary Geometry, but simply as an edition of a part of the Syllabus with the demonstrations supplied and suitable exercises inserted.

Probably many teachers will find in this all that they require in a Text Book for their pupils, being satisfied to supply the needful illustrations, explanations, and developments of the subject in their oral teaching. Still there is, doubtless, room for other treatises embodying such illustrative and explanatory matter; but the Association is of opinion that such treatises should rather be the work of individual authors than of an Association.

Accordingly, should any author desire to publish a further treatise, based on the present work, the Council of the Association would be glad to authorise his free use of the work on terms to be arranged by communication with them through the publishers or the honorary secretaries.

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SYLLABUS

OF

GEOMETRICAL CONSTRUCTIONS.

[NOTE.—The Association recommend that beginners, prior to or concurrently with the study of Theoretical Geometry, should be exercised in simple Geometrical Drawing, in order to familiarize them with the conceptions, and give them some general notions of the nature of the results, of the Science of Geometry. With this object the following Syllabus of Constructions has been drawn up.]

THE following constructions are to be made with the Ruler and Compasses only; the ruler being used for drawing and producing straight lines, the compasses for describing circles and for the transference of distances.

1. The bisection of an angle.
2. The bisection of a straight line.
3. The drawing of a perpendicular at a point in, and from a point outside, a given straight line, and the determination of the projection of a finite line on a given straight line.
4. The construction of an angle equal to a given angle; of an angle equal to the sum of two given angles, etc.
5. The drawing of a line parallel to another under various conditions—and hence the division of lines into aliquot parts, in given ratio, etc.

6. The construction of a triangle, having given
(α) three sides;
(β) two sides and contained angle;
(γ) two angles and side adjacent;
(δ) two angles and side opposite.

7. The drawing of tangents to circles, under various conditions.

8. The inscription and circumscription of figures in and about circles; and of circles in and about figures.

7 and 8 may be deferred till the Straight Line and Triangles have been studied theoretically, but should in all cases precede the study of the Circle.

The above constructions are to be taught *generally*, and illustrated by one or more of the following classes of problems:

(α) The making of constructions involving various combinations of the above in accordance with *general* (*i.e.*, not *numerical*) conditions, and exhibiting some of the more remarkable results of Geometry, such as the circumstances under which more than two straight lines pass through a point, or more than two points lie on a straight line.

(β) The making of the above constructions and combinations of them to scale (but without the protractor).

(γ) The application of the above constructions to the indirect measurement of distances.

(δ) The use of the protractor and scale of chords, and the application of these to the laying off of angles, and the indirect measurement of angles.

INTRODUCTION.

[NOTE.—In the following Introduction are collected together certain general axioms which, though frequently used in Geometry, are not peculiar to that science, and also certain logical relations, the distinct apprehension of which is very desirable in connection with the demonstrations of the Propositions. They are brought together here for convenience of reference, but it is not intended to imply by this that the study of Geometry ought to be preceded by a study of the logical interdependence of associated theorems. The Association think that at first all the steps by which any theorem is demonstrated should be carefully gone through by the student, rather than that its truth should be inferred from the logical rules here laid down. At the same time they strongly recommend an early application of general logical principles.]

1. Propositions admitted without demonstration are called *Axioms*.

2. Of the Axioms used in Geometry those are termed *General* which are applicable to magnitudes of all kinds: the following is a list of certain general axioms frequently used.

(*a*) The whole is greater than its part.

(*b*) The whole is equal to the sum of its parts.

(*c*) Things that are equal to the same thing are equal to one another.

(*d*) If equals are added to equals the sums are equal.

(*e*) If equals are taken from equals the remainders are equal.

(*f*) If equals are added to unequals the sums are unequal, the greater sum being that which is obtained from the greater magnitude.

(*g*) If equals are taken from unequals the remainders are unequal, the greater remainder being that which is obtained from the greater magnitude.

(*h*) The halves of equals are equal.

3. A *Theorem* is a proposition enunciating a fact whose truth is demonstrated from known propositions. These known propositions may themselves be Theorems or Axioms.

4. The enunciation of a Theorem consists of two parts,—the *hypothesis*, or that which is assumed, and the *conclusion*, or that which is asserted to follow therefrom. Thus in the typical Theorem,

If A is B, then C is D, (i),

the hypothesis is that A is B, and the conclusion, that C is D.

From this Theorem it necessarily follows that:

If C is not D, then A is not B, (ii).

Two such Theorems as (i) and (ii) are said to be *contrapositive*, each of the other.

5. Two Theorems are said to be *converse*, each of the other, when the hypothesis of each is the conclusion of the other.

Thus,

If C is D, then A is B, (iii)

is the converse of the typical Theorem (i).

The contrapositive of the last Theorem, viz.:

If A is not B, then C is not D, (iv)

is termed the *obverse* of the typical Theorem (i).

6. Sometimes the hypothesis of a Theorem is complex, *i.e.*, consists of several distinct hypotheses; in this case every Theorem formed by interchanging the conclusion and one of the hypotheses is a converse of the original Theorem.

7 The truth of a converse is not a logical consequence of the truth of the original Theorem, but requires independent investigation.

8. Hence the four associated Theorems (i) (ii) (iii) (iv) resolve themselves into two Theorems that are independent of one another, and two others that are always and necessarily true if the former are true; consequently it will never be necessary to demonstrate geometrically more than two of the four Theorems, care being taken that the two selected are not contrapositive each of the other.

9. *Rule of Conversion.* If the hypotheses of a group of demonstrated Theorems be exhaustive—that is, form a set of alternatives of which one must be true; and if the conclusions be mutually exclusive—that is, be such that no two of them can be true at the same time, then the converse of every Theorem of the group will necessarily be true.

The simplest example of such a group is presented when a Theorem and its obverse have been demonstrated; and the validity of the rule in this instance is obvious from the circumstance that the converse of each of two such Theorems is the contrapositive of the other. Another example, of frequent occurrence in the elements of Geometry, is of the following type :

If A is greater than B, C is greater than D.
If A is equal to B, C is equal to D.
If A is less than B, C is less than D.

Three such Theorems having been demonstrated the converse of each is necessarily true.

10. *Rule of Identity.* If there is but one A, and but one B; then from the fact that A is B it necessarily follows that B is A.

This rule may be frequently applied with great advantage in the demonstration of the converse of an established Theorem.

THE ELEMENTS OF PLANE GEOMETRY.

BOOK I.

THE STRAIGHT LINE.

DEFINITIONS.

DEF. 1. *A* **point** *has position, but it has no magnitude.*

DEF. 2. *A* **line** *has position, and it has length, but neither breadth nor thickness.*

The extremities of a line are points, and the intersection of two lines is a point.

DEF. 3. *A* **surface** *has position, and it has length and breadth, but not thickness.*

The boundaries of a surface, and the intersection of two surfaces, are lines.

DEF. 4. *A* **solid** *has position, and it has length, breadth and thickness.*

The boundaries of a solid are surfaces.

DEF. 5. *A* **straight line** *is such that any part will, however placed, lie wholly on any other part, if its extremities are made to fall on that other part.*

DEF. 6. *A* **plane** *surface, or plane, is a surface in which any two points being taken the straight line that joins them lies wholly in that surface.*

GEOMETRICAL AXIOMS.

1. Magnitudes that can be made to coincide are equal.

2. Through two points there can be made to pass one, and only one, straight line: and this may be indefinitely prolonged either way.

Hence,

α. Any straight line may be made to fall on any other straight line with any given point on the one on any given point on the other;

β. Two straight lines which meet in one point cannot meet again unless they coincide.

SECTION I.

ANGLES AT A POINT.

[An *angle* is a simple concept incapable of definition, properly so-called, but the nature of the concept may be explained as follows, and for convenience of reference the explanation may be reckoned among the definitions.]

DEF. 7. *When two straight lines are drawn from the same point, they are said to contain, or to make with each other, a* **plane angle.** The point is called the *vertex*, and the straight lines are called the *arms*, of the angle.

A line drawn from the vertex and turning about the vertex in the plane of the angle from the position of coincidence with one arm to that of coincidence with the other is said to turn through the angle: and the angle is greater as the quantity of turning is greater.

Since the line may turn from the one position to the other in either of two ways, two angles are formed by two straight lines drawn from a point. These angles (which have a common vertex and common arms) are said to be *conjugate.* The greater of the two is called the *major conjugate*, and the smaller the *minor conjugate*, angle.

When *the angle contained by two lines* is spoken of without qualification, the *minor conjugate* angle is to be understood. It is seldom requisite to consider major conjugate angles before Book III.

When the arms of an angle are in the same straight line, the conjugate angles are equal, and each is then said to be
7 *a* **straight angle.**

Def. 8. *When three straight lines are drawn from a point, if one of them be regarded as lying between the other two, the angles which this one (the mean) makes with the other two (the extremes) are said to be* **adjacent** *angles: and the angle between the extremes, through which a line would turn in passing from one extreme through the mean to the other extreme, is the sum of the two adjacent angles.*

Def. 9. *The* **bisector** *of an angle is the straight line that divides it into two equal angles.*

Def. 10. *When one straight line stands upon another straight line and makes the adjacent angles equal, each of the angles is called a* **right angle.**

Obs. Hence a straight angle is equal to two right angles; or, a right angle is half a straight angle.

Def. 11. *A* **perpendicular** *to a straight line is a straight line that makes a right angle with it.*

DEF. 12. *An* **acute** *angle is that which is less than a right angle.*

DEF. 13. *An* **obtuse** *angle is that which is greater than one right angle, but less than two right angles.*

DEF. 14. *A* **reflex** *angle is a term sometimes used for a major conjugate angle.*

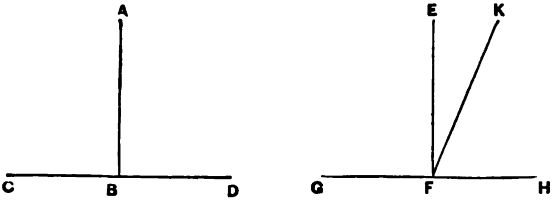
DEF. 15. *When the sum of two angles is a right angle, each is called the* **complement** *of the other, or is said to be* **complementary** *to the other.*

DEF. 16. *When the sum of two angles is two right angles, each is called the* **supplement** *of the other, or is said to be* **supplementary** *to the other.*

DEF. 17. *The opposite angles made by two straight lines that intersect are called* **vertically opposite angles.**

THEOR. 1. All right angles are equal to one another.

Let ABC be a right angle formed by the straight line AB standing on the straight line CBD,
EFG a right angle formed by the straight line EF standing on the straight line GFH:



then shall the angle ABC be equal to thc angle EFG.

Apply the straight line CBD to the straight line GFH,
so that the point B may fall on the point F, *Ax. 2a.*

and the straight line BA on the same side of GFH as FE;
the line BA shall fall on the line FE.
For if it falls otherwise it must fall either within the angle EFH or the angle EFG.

Let it fall within the angle EFH, as FK.
Then, because the angle KFG is a right angle,
therefore it is equal to the angle KFH; *Def.* 10.
but the angle EFH is greater than the angle KFH, *Ax. a.*
therefore the angle EFH is also greater than the angle KFG,
much more then is the angle EFH greater than
the angle EFG. *Ax. a.*
But because the angle EFG is a right angle, therefore it is equal to the angle EFH; *Def.* 10.
therefore the angle EFG is both greater than, and equal to, the angle EFH,
which is impossible,
therefore the line BA does not fall within the angle EFH.

In the same way it may be proved that the line BA does not fall within the angle EFG.

Hence the line BA does fall on the line FE.
Therefore the angle ABC coincides with the angle EFG,
and therefore the angle ABC is equal to the angle EFG. *Ax.* 1.

Q.E.D.

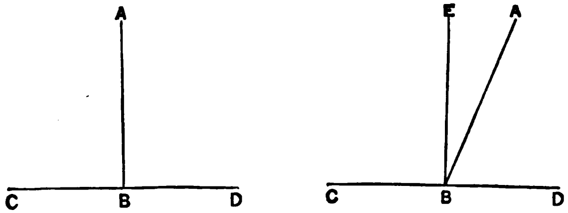
COR. 1. At a given point in a given straight line only one perpendicular can be drawn to that line.

COR. 2. The complements of equal angles are equal.

COR. 3. The supplements of equal angles are equal.

THEOR. 2. If a straight line stands upon another straight line, it makes the adjacent angles together equal to two right angles.

Let the straight line AB stand upon the straight line CD:



then shall the angles ABC, ABD be together equal to two right angles.

If the angle ABC is equal to the angle ABD, each of them is a right angle, *Def.* 10.
and therefore they are together equal to two right angles.

But if the angle ABC is not equal to the angle ABD,
let BE be a straight line standing upon the straight line CBD, so as to make the angle CBE equal to the angle EBD,
then the angles CBE, EBD are two right angles. *Def.* 10.
Now the angle ABC is equal to the two angles CBE, EBA, *Ax. b.*
to each of these equals add the angle ABD,
then the two angles ABC, ABD are together equal to the three angles CBE, EBA, ABD. *Ax. d.*
Again, the angle EBD is equal to the two angles EBA, ABD,
to each of these equals add the angle EBC,
then the two angles EBD, EBC are together equal to the three angles CBE, EBA, ABD; *Ax. d.*

therefore the angles ABC, ABD are together equal to the angles EBC, EBD; *Ax. c.*
therefore the angles ABC, ABD are together equal to two right angles.

Q.E.D.

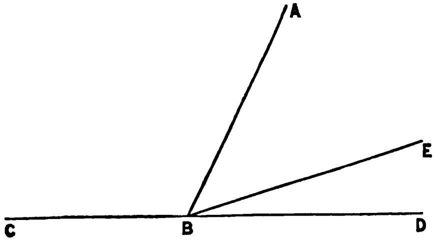
COR. All the angles made by any number of straight lines drawn from a point, each with the next following in order, are together equal to four right angles.

Ex. 1. If two straight lines intersect, and one of the four angles so formed is a right angle, shew that the other three angles are also right angles.

**Ex.* 2. The bisectors of the adjacent angles which one straight line makes with another include a right angle.

THEOR. 3. If the adjacent angles made by one straight line with two others are together equal to two right angles, these two straight lines are in one straight line.

Let the adjacent angles ABC, ABD which the straight line AB makes with the two other straight lines BC, BD be together equal to two right angles:



then shall BC and BD be in one straight line.

---

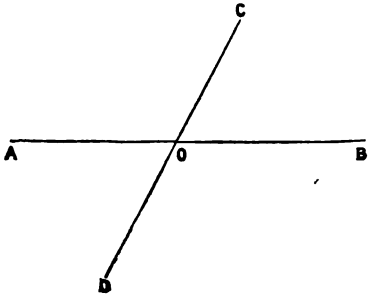
* Exercises marked with an asterisk are worth remembering as results, or with a view to the solution of subsequent exercises.

For if BD is not in a straight line with BC,
let BE be in a straight line with BC.

Then because AB stands on the straight line CBE,
therefore the angles ABC, ABE are together equal to two right angles; *I. 2.*
but the angles ABC, ABD are also together equal to two right angles; *Hyp.*
therefore the angles ABC, ABE are together equal to the angles ABC, ABD. *Ax. c.*
Take away the common angle ABC;
then the angle ABE is equal to the angle ABD, *Ax. e.*
the part to the whole,
which is impossible; *Ax. a.*
therefore BE is not in a straight line with BC.
In the same way it can be shown that no other straight line than BD is in a straight line with BC,
therefore BC and BD are in one straight line. Q.E.D.

THEOR. 4. If two straight lines cut one another, the vertically opposite angles are equal to one another.

Let the two straight lines AB, CD cut one another at O:



then shall the angle AOC be equal to the angle BOD, and the angle BOC to the angle AOD.

Because AO stands upon CD,
therefore the angles AOC, AOD are together equal to two right angles; *I.* 2.
again, because DO stands upon AB,
therefore the angles AOD, DOB are together equal to two right angles; *I.* 2.
therefore the angles AOC, AOD are together equal to the angles AOD, DOB; *Ax. c.*
take away the common angle AOD;
then the angle AOC is equal to the angle BOD. *Ax. e.*

In the same way it may be proved that the angle BOC is equal to the angle AOD. Q.E.D.

Ex. 3. The bisectors of two vertically opposite angles are in one straight line.

SECTION II.

TRIANGLES.

DEF. 18. *A plane* **figure** *is a portion of a plane surface inclosed by a line or lines.*

DEF. 19. *Figures that may be made by superposition to coincide with one another are said to be* **identically** *equal; or they are said to be equal in all respects.*

DEF. 20. *The* **area** *of a plane figure is the quantity of the plane surface inclosed by its boundary.*

DEF. 21. *A* **plane rectilineal figure** *is a portion of a plane surface inclosed by straight lines. When there are more than three inclosing straight lines the figure is called a* **polygon.**

DEF. 22. *A polygon is said to be* **convex** *when no one of its angles is reflex.*

DEF. 23. *A polygon is said to be* **regular** *when it is equilateral and equiangular; that is, when its sides and angles are equal.*

DEF. 24. *A* **diagonal** *is the straight line joining the vertices of any angles of a polygon which have not a common arm.*

DEF. 25. *The* **perimeter** *of a rectilineal figure is the sum of its sides.*

DEF. 26. *A* **quadrilateral** *is a polygon of four sides, a* **pentagon** *one of five sides, a* **hexagon** *one of six sides, and so on.*

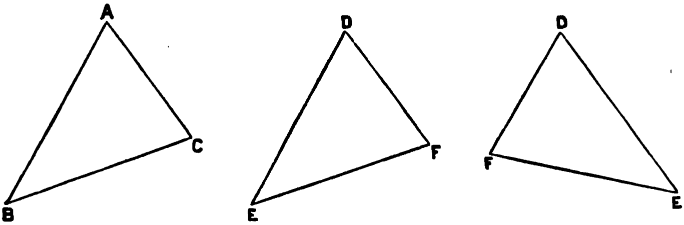
DEF. 27. *A* **triangle** *is a figure contained by three straight lines.*

DEF. 28. *Any side of a triangle may be called the* **base,** *and the opposite angular point is then called the* **vertex.**

DEF. 29. *An* **isosceles** *triangle is that which has two sides equal; the angle contained by those sides is called the* **vertical angle**, *the third side the* **base**.

THEOR. 5. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles included by these sides equal, then the triangles are identically equal, and of the angles those are equal which are opposite to the equal sides.

Let ABC, DEF be two triangles having the side AB equal to the side DE, the side AC to the side DF, and the angle BAC to the angle EDF :



then shall the triangles be identically equal, having the side BC equal to the side EF, the angle ACB to the angle DFE, and the angle ABC to the angle DEF.

Let the triangle ABC be applied to the triangle DEF, so that the point A may fall on the point D, the side AB along the side DE, and the point C on the same side of DE as the point F ;
then B will fall on E, since AB is equal to DE, *Hyp.*

AC will fall along DF, since the angle BAC is equal to the angle EDF, *Hyp.*
and, AC falling along DF,
C will fall on F, since AC is equal to DF. *Hyp.*

Hence, B falling on E, and C on F, BC will coincide with EF, *Ax.* 2.
and the triangle ABC will coincide with the triangle DEF,
and is therefore identically equal to it, *Ax.* 1.
the side BC equal to the side EF, the angle ACB to the angle DFE, and the angle ABC to the angle DEF.

Q.E.D.

Ex. 4. The straight line which bisects the vertical angle of an isosceles triangle bisects the base.

Ex. 5. Any point on the bisector of the vertical angle of an isosceles triangle is equidistant from the extremities of the base.

Ex. 6. The straight line which bisects the vertical angle of an isosceles triangle is perpendicular to the base.

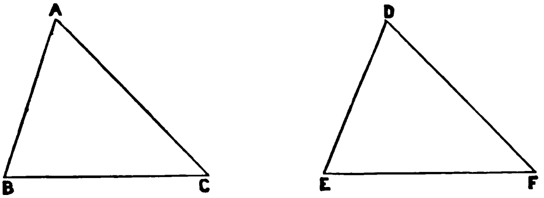
Ex. 7. Any point D is taken on the bisector of an angle BAC; prove that, if AB is equal to AC, then the angle ADB is equal to the angle ADC.

Ex. 8. The straight lines drawn from the extremities of the base of an isosceles triangle to the middle points of the opposite sides are equal to one another.

Ex. 9. On one arm of an angle whose vertex is A points B and D are taken, and on the other arm points C and E, such that AB is equal to AC, and AD to AE: shew that BE is equal to CD.

THEOR. 6. If two triangles have two angles of the one equal to two angles of the other, each to each, and have likewise the sides between the vertices of these angles equal, then the triangles are identically equal, and of the sides those are equal which are opposite to the equal angles.

Let ABC, DEF be two triangles having the angle ABC equal to the angle DEF, the angle ACB to the angle DFE, and the side BC to the side EF:



then shall the triangles be identically equal, having the angle BAC equal to the angle EDF, the side AC to the side DF, and the side AB to the side DE.

Let the triangle ABC be applied to the triangle DEF, so that the point B may fall on the point E, the side BC along the side EF, and the point A on the same side of EF as the point D;
then C will fall on F, since BC is equal to EF, *Hyp.*
BA will fall along ED, since the angle CBA is equal to the angle FED, *Hyp.*
and CA will fall along FD, since the angle BCA is equal to the angle EFD; *Hyp.*
hence A, which is the point of intersection of BA and CA, will fall on D, which is the point of intersection of ED and FD,

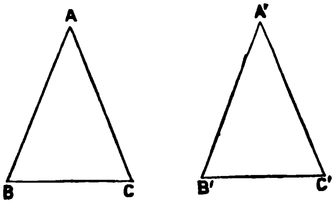
and the triangle ABC will coincide with the triangle DEF, and is therefore identically equal to it, *Ax.* 1. the angle BAC equal to the angle EDF, the side AC to the side DF, and the side AB to the side DE.

Q.E.D.

Ex. 10. If the bisector of an angle of a triangle is also perpendicular to the opposite side, the triangle is isosceles.

THEOR. 7. If two sides of a triangle are equal, the angles opposite to those sides are equal.

Let ABC be a triangle having the side AB equal to the side AC:



then shall the angle ACB be equal to the angle ABC.

Let A′B′C′ be a triangle identically equal to the triangle ABC, the points A′,B′,C′ corresponding respectively to the points A,B,C.

Then in the triangles ABC, A′C′B′,
AB is equal to A′C′, since it is equal to AC, *Hyp.*
and AC is equal to A′B′, since it is equal to AB, *Hyp.*
and the angle BAC is equal to the angle C′A′B′,

therefore the angle ACB, which is opposite to the side AB, is equal to the angle A′B′C′, which is opposite to the side A′C′, *I.* 5. that is, the angle ACB is equal to the angle ABC.

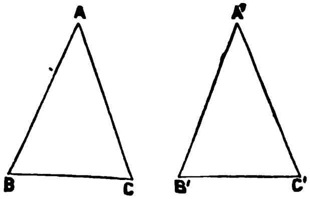
Q.E.D.

COR. If a triangle is equilateral, it is also equiangular.

Ex. 11. Prove Theor. 7 by comparing the triangles into which the bisector of the vertical angle divides the isosceles triangle.

THEOR. 8. If two angles of a triangle are equal, the sides opposite to those angles are equal.

Let ABC be a triangle having the angle ABC equal to the angle ACB:



then shall the side AC be equal to the side AB.

Let A′B′C′ be a triangle identically equal to the triangle ABC, the points A′,B′,C′ corresponding respectively to the points A,B,C.

Then in the triangles ABC, A′C′B′,
the angle ABC is equal to the angle A′C′B′,
since it is equal to the angle ACB, *Hyp.*

and the angle ACB is equal to the angle A′B′C′,
since it is equal to the angle ABC, *Hyp.*
and BC is equal to C′B′,
therefore the side AC, which is opposite to the angle ABC, is equal to the side A′B′, which is opposite to the angle A′C′B′, *I.* 6.
that is, the side AC is equal to the side AB.

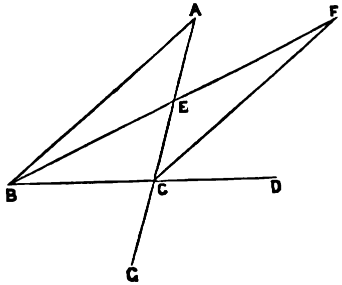
Q.E.D.

COR. If a triangle is equiangular, it is also equilateral.

Ex. 12. If the angles at the base of an isosceles triangle are bisected, the bisectors and the base form an isosceles triangle.

THEOR. 9. If any side of a triangle is produced, the exterior angle is greater than either of the interior opposite angles.

Let ABC be a triangle having the side BC produced to D:



then shall the exterior angle ACD be greater than either of the interior opposite angles BAC, ABC.

Let E be the middle point of AC;
join BE, and produce BE to F, making EF equal to BE,
join FC.

Then in the triangles ECF, EAB,
the side EC is equal to the side EA,
the side EF is equal to the side EB,
and the angle CEF is equal to the angle AEB,
since they are vertically opposite, *I.* 4.
therefore the angle ECF is equal to the angle EAB, *I.* 5.
but the angle ACD is greater than the angle ECF, *Ax. a.*
therefore the angle ACD is also greater than the angle BAC;
which is the angle opposite to the side BC which is produced.

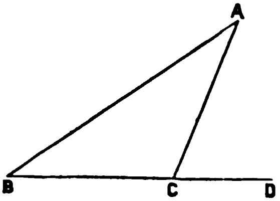
Again, produce AC to G,
then, in like manner, the angle BCG is greater than the angle ABC which is opposite to AC,
but the angle BCG is equal to the angle ACD,
because they are vertically opposite; *I.* 4.
therefore the angle ACD is also greater than the angle ABC.

Q.E.D.

Ex. 13. Shew that the sum of the angles of the triangle FBC is equal to the sum of the angles of the triangle ABC.

THEOR. 10. Any two angles of a triangle are together less than two right angles.

Let ABC be a triangle:



then shall any two of its angles be together less than two right angles.

Produce BC to D.

Because ACD is an exterior angle of the triangle ABC, therefore it is greater than the interior opposite angle ABC, *I.* 9.
to each add the angle ACB,
then the angles ACD, ACB are together greater than the angles ABC, ACB ; *Ax. f.*
but the angles ACD, ACB are together equal to two right angles, *I.* 2.
therefore the angles ABC, ACB are together less than two right angles.

In the same way it may be shown that the angles BCA, CAB, or the angles CAB, ABC are together less than two right angles.

Q.E.D.

COR. 1. If a triangle has one right angle or obtuse angle, its remaining angles are acute.

COR. 2. Every triangle has at least two acute angles.

Hence,

DEF. 30. *A triangle which has one of its angles a right angle is called a* **right-angled** *triangle. A triangle which has one of its angles an obtuse angle is called an* **obtuse-angled** *triangle. A triangle which has all its angles acute is called an* **acute-angled** *triangle.*

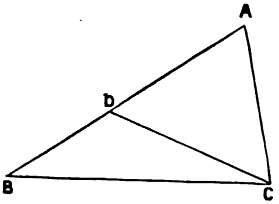
DEF. 31. *The side of a right-angled triangle which is opposite to the right-angle is called the* **hypotenuse.**

COR. 3. From a given point outside a given straight line, only one perpendicular can be drawn to that line.

Ex. 14. Prove Theor. 10 by joining the vertex to any point in the base, and using Theor. 9 twice.

THEOR. 11. If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.

Let ABC be a triangle having the side AB greater than the side AC :



then shall the angle ACB be greater than the angle ABC.

From AB cut off AD equal to AC, join CD.

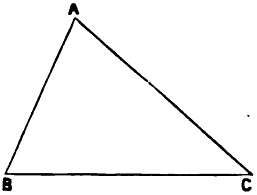
Then because AD is equal to AC,
therefore the angle ADC is equal to the angle ACD ; *I.* 7.
but the angle ADC, being an exterior angle of the triangle BDC,
is greater than the interior opposite angle ABC, *I.* 9.
therefore the angle ACD is greater than the angle ABC ;
still more then is the angle ACB greater than the angle ABC.

Q.E.D.

Ex. 15. Prove Theor. 8 by means of Theor. 11.

THEOR 12. If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.

Let ABC be a triangle having the angle ABC greater than the angle ACB :



then shall the side AC be greater than the side AB.

If AC is not greater than AB,
then it is either equal to, or less than, AB.
But AC is not equal to AB,
for then the angle ABC would be equal to the angle ACB ; *I.* 7.
also AC is not less than AB,
for then the angle ABC would be less than the angle ACB ; *I.* 11.
therefore AC is greater than AB.

Q.E.D.

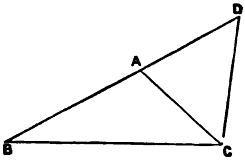
NOTE.—As to this proof and that of Ex. 15 *cf.* Introduction, § 9.

Ex. 16. The hypotenuse of a right-angled triangle is greater than either of the remaining sides.

Ex. 17. In an obtuse-angled triangle the side opposite the obtuse angle is the greatest.

THEOR. 13. Any two sides of a triangle are together greater than the third side.

Let ABC be a triangle:



then shall the sides BA and AC be together greater than the side BC, AC and CB than AB, and CB and BA than AC.

Produce BA to D, make AD equal to AC, and join CD.

Because AD is equal to AC,
therefore the angle ACD is equal to the angle ADC; *I.* 7.
but the angle BCD is greater than the angle ACD, *Ax. a.*
therefore the angle BCD is also greater than the angle ADC, that is, than the angle BDC,
therefore the side BD of the triangle BDC is greater than the side BC; *I.* 12.
but BA and AC are together equal to BD,
since AC is equal to AD,
therefore BA and AC are together greater than BC.

Similarly it may be shown that AC and CB are together greater than AB, and CB and BA than AC.

Q.E.D.

COR. The difference of any two sides of a triangle is less than the third side.

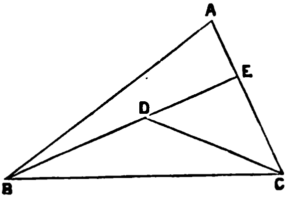
**Ex.* 18. The straight line drawn from the vertex of a triangle to the middle point of the base is less than half the sum of the remaining sides. Use the construction of Theor. 9.

Ex. 19. If O is a point within the triangle ABC, shew that the sum of OA, OB, and OC is greater than half the perimeter of the triangle.

Ex. 20. The perimeter of a quadrilateral is greater than the sum, and less than twice the sum of the diagonals.

THEOR. 14. If from the ends of a side of a triangle two straight lines are drawn to a point within the triangle, these are together less than the two other sides of the triangle, but contain a greater angle.

Let ABC be a triangle, and from the ends of a side BC let straight lines BD, CD be drawn to a point D within the triangle :



then shall BD and DC be together less than BA and AC, but the angle BDC shall be greater than the angle BAC.

Produce BD to meet AC at E.

Then BA and AE are together greater than BE, *I.* 13.
to each of these add EC,
then BA and AC are together greater than BE and EC ; *Ax. f.*

again, DE and EC are together greater than DC, *I.* 13.
to each of these add DB,
then BE and EC are together greater than BD and DC; *Ax. f.*
but BA and AC are together greater than BE and EC,
still more then are BA and AC together greater than BD and DC.

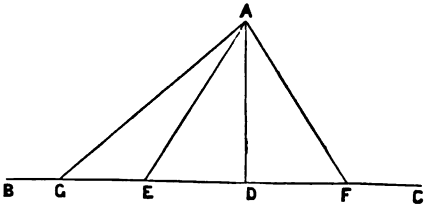
Again, the angle BDC, being an exterior angle of the triangle CED, is greater than the interior opposite angle DEC; *I.* 9.
and the angle DEC, being an exterior angle of the triangle BAE, is greater than the angle BAC,
still more then is the angle BDC greater than the angle BAC.

Q.E.D.

Ex. 21. If a point O be taken within the triangle ABC, the sum of OA, OB, and OC is less than the perimeter of the triangle.

THEOR. 15. Of all the straight lines that can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest; and of the others, those which make equal angles with the perpendicular are equal; and that which makes a greater angle with the perpendicular is greater than that which makes a less angle.

Let A be the given point, and BC the given straight line,
and let AD be the perpendicular from A to BC,
and let AE, AF be any straight lines making equal angles EAD, FAD with AD,
and let AG make with AD the angle GAD greater than the angle EAD, or FAD:



then shall AD be less than AE,
and AF equal to AE,
and AG greater than AE or AF.

In the triangle ADE,
the angles ADE and AED are less than two right angles; *I.* 10.
but ADE is a right angle *Hyp.*
therefore AED is less than a right angle,
therefore AED is less than ADE,
and therefore AD is less than AE. *I.* 12.

Again, in the triangles ADE, ADF,
the angle ADE is equal to the angle ADF,
since they are both right angles,
the angle EAD is equal to the angle FAD, *Hyp.*
and the side AD is common to both,
therefore AE is equal to AF. *I.* 6.

Also, of AE and AF let AE be the one that is on the same side of the perpendicular as AG,
then, in the triangle AGE,
the angle AEG, being an exterior angle of the triangle AED, is greater than the interior opposite angle ADE, *I.* 9.

and is therefore an obtuse angle,
and therefore AGE is an acute angle, *I.* 10.
and AEG is greater than AGE,
therefore AG is greater than AE, *I.* 12.
and therefore is also greater than AF.

Q.E.D.

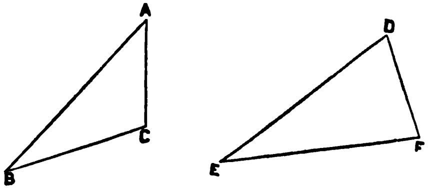
COR. Not more than two equal straight lines can be drawn from a given point to a given straight line.

DEF. 32. *The perpendicular to a given straight line from a given point outside it is called the* **distance** *of the point from the straight line.*

Ex. 22. Shew from Theor. 15 that the hypotenuse is the greatest side of a right-angled triangle.

THEOR. 16. If two triangles have two sides of the one equal to two sides of the other, each to each, but the included angles unequal, then the bases are unequal, the base of that which has the greater angle being greater than the base of the other.

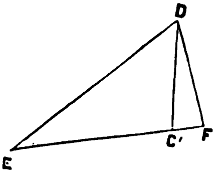
Let ABC, DEF be two triangles having the side AB equal to the side DE, and the side AC to the side DF, but the angles BAC, EDF unequal, EDF being greater than BAC:



then shall the base EF be greater than the base BC.

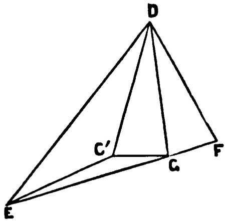
Apply the triangle ABC to the triangle DEF, so that A may fall on D, and AB along DE, and the point C at C′ on the same side of DE as the point F,
then B will fall on E,
because AB is equal to DE. *Hyp.*

If C′ is in the straight line EF, it must lie between E and F,



because the angle EDF is greater than the angle EDC′,
and therefore EC′, that is BC, is less than EF. *Ax. a.*

But if C′ is not in the straight line EF,



let the line that bisects the angle FDC′ meet EF at G,
join C′G.

Then, in the triangles C′DG, FDG,
the side C′D is equal to the side DF,
the side DG is common to both,

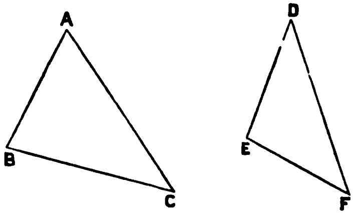
and the angle C′DG is equal to the angle FDG,
therefore C′G is equal to GF, *I.* 5.
therefore EG and GC′ are together equal to EF; *Ax. d.*
but EG and GC′ are together greater than EC′, *I.* 13.
therefore EF is greater than EC′,
that is, than BC.

Q.E.D.

Ex. 23. If D is the middle point of the base BC of the triangle ABC, and the angle ADB is obtuse, shew that AB is greater than AC.

THEOR. 17. If two triangles have two sides of the one equal to two sides of the other, each to each, but the bases unequal, then the included angles are unequal, the angle of that which has the greater base being greater than the angle of the other.

Let ABC, DEF be two triangles having the side AB equal to the side DE, and the side AC equal to the side DF, but the base BC greater than the base EF:



then shall the angle BAC be greater than the angle EDF.

If the angle BAC is not greater than the angle EDF, it is either equal to, or less than, the angle EDF.

But the angle BAC is not equal to the angle EDF,

for then the base BC would be equal to the base EF; *I.* 5.
also the angle BAC is not less than the angle EDF,
for then the base BC would be less than the base EF; *I.* 16.
therefore the angle BAC is greater than the angle EDF.

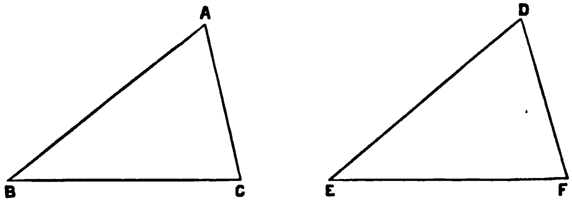
Q.E.D.

NOTE.—*Cf.* Introduction, § 9.

Ex. 24. If in the triangle ABC, AB is less than AC and AD is drawn to D the middle point of BC, shew that the angle ADB is acute.

THEOR. 18. If two triangles have the three sides of the one equal to the three sides of the other, each to each, then the triangles are identically equal, and of the angles those are equal which are opposite to equal sides.

Let ABC, DEF be two triangles having the side BC equal to the side EF, the side CA to the side FD, and the side AB to the side DE:



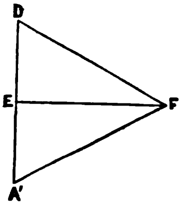
then shall the triangles be identically equal, the angle CAB equal to the angle FDE, the angle ABC to the angle DEF, and the angle BCA to the angle EFD.

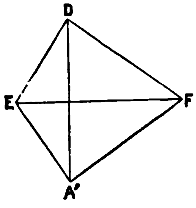
First Proof. In the triangles ABC, DEF,
the side AB is equal to the side DE, *Hyp.*

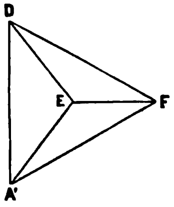
and the side AC is equal to the side DF, *Hyp.*
if, further, the angle BAC were unequal to the angle EDF,
then the base BC would be unequal to the base EF, *I.* 16.
therefore the angle BAC is not unequal to the angle EDF, that is, it is equal to it; *Hyp.*
therefore the triangles are identically equal, *I.* 5.
the angle ABC equal to the angle DEF, and the angle BCA to the angle EFD.

Q.E.D.

Second Proof. Let the triangle ABC be applied to the triangle DEF, so that B may fall on E, and BC along EF, and the point A on the opposite side of EF to the point D,







then C will fall on F,
because BC is equal to EF. *Hyp.*

Let A′ be the point where A falls.

If DEA′ is a straight line,
then the angle FDE is equal to the angle FA′E. *I.* 7.

But if DEA′ is not a straight line,
join DA′.

Then, because FD is equal to FA′, *Hyp.*

therefore the angle FDA′ is equal to the angle FA′D; *I.* 7.
and, because ED is equal to EA′,
therefore the angle EDA′ is equal to the angle EA′D; *I.* 7.
therefore the whole, or remaining, angle FDE is equal to the whole, or remaining, angle FA′E. *Ax. d, or e.*

Therefore, in all cases, the angle FDE is equal to the angle FA′E, that is, to the angle CAB.

Hence, in the triangles ABC, DEF,
the side AB is equal to the side DE, *Hyp.*
the side AC is equal to the side DF, *Hyp.*
and the angle BAC is equal to the angle EDF,
therefore the triangles are identically equal, *I.* 5.
the angle ABC equal to the angle DEF, and the angle ACB to the angle DFE.

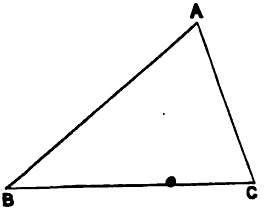
Q.E.D.

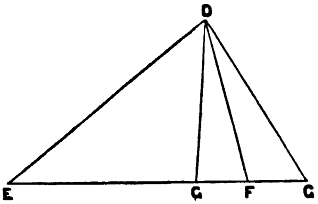
Ex. 25. If the opposite sides of a quadrilateral are equal, shew that the opposite angles are also equal.

Ex. 26. Two isosceles triangles are on the same base and on opposite sides of it; shew that the straight line joining their vertices bisects the vertical angles.

THEOR. 19. If two triangles have two angles of the one equal to two angles of the other, each to each, and have likewise the sides opposite to one pair of equal angles equal, then the triangles are identically equal, and of the sides those are equal which are opposite to equal angles.

Let ABC, DEF be two triangles having the angle ABC equal to the angle DEF, the angle BCA equal to the angle EFD, and the side AB equal to the side DE:





then shall the triangles be identically equal,
the angle BAC equal to the angle EDF, the side BC to the side EF, and the side CA to the side FD.

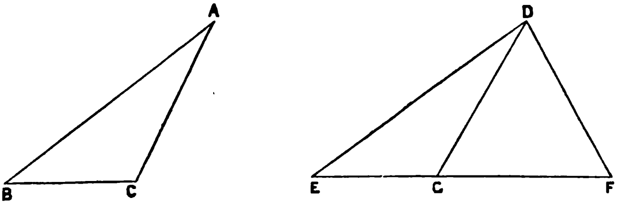
Apply the triangle ABC to the triangle DEF, so that A may fall on D, and AB along DE, and the point C on the same side of DE as the point F,
then B will fall on E,
because AB is equal to DE, *Hyp.*
and BC will fall along EF,
because the angle ABC is equal to the angle DEF. *Hyp.*
Then if C do not fall on F,
it will fall on some other point G in EF or EF produced;
then, of the two angles DFE and DGE, the one is an exterior, and the other an interior opposite angle, of the triangle DFG,
therefore the angle DFE is not equal to the angle DGE, *I.* 9.
that is, to the angle ACB;
but it is equal to the angle ACB; *Hyp.*
hence C does not fall otherwise than on F.
Therefore the two triangles coincide,
and therefore are identically equal, *Def.* 30.
the angle BAC equal to the angle EDF, the side BC to the side EF, and the side CA to the side FD.

Q.E.D.

Ex. 27. Shew that any point in the bisector of an angle is equidistant from the arms of the angle.

THEOR. 20. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles opposite to one pair of equal sides equal, then the angles opposite to the other pair of equal sides are either equal or supplementary, and in the former case the triangles are identically equal.

Let ABC, DEF be two triangles having the side AB equal to the side DE, the side AC to the side DF, and the angle ABC to the angle DEF:



then shall the angles ACB, DFE be either equal or supplementary, and, in the former case, the triangles shall be identically equal.

Apply the triangle ABC to the triangle DEF, so that A may fall on D, and AB along DE, and the point C on the same side of DE as the point F,

then B will fall on E,

since AB is equal to DE, *Hyp.*

and BC will fall along EF,

since the angle ABC is equal to the angle DEF, *Hyp.*

therefore C will fall on F,
or in EF or EF produced.

If C falls on F,
the triangles coincide,
and therefore are identically equal. *Def.* 30.

If C falls in EF, or EF produced, as at G,
then because DG is equal to DF,
therefore the angle DFG is equal to the angle DGF, *I.* 7.
therefore the angles DGE, DFE are supplementary, *I.* 2.
that is, the angles ACB, DFE are supplementary.

Q.E.D.

COR. Two such triangles are identically equal

1. **If the two angles given equal are right angles or obtuse angles.**
2. **If the angles opposite to the other two equal sides are both acute, or both obtuse, or if one of them is a right angle.**
3. **If the side opposite the given angle in each triangle is not less than the other given side.**

Ex. 28. The point O is equidistant from the arms of the angle BAC, shew that OA bisects the angle BAC.

EXERCISES.

29. If two quadrilaterals have three sides of the one equal respectively to three sides of the other taken in order, and have likewise the angles contained by those sides equal to one another, each to each, they are equal in all respects.

30. Two isosceles triangles are on the same base; shew that the straight line through their vertices bisects the base at right angles.

31. In the equal sides AB, AC of an isosceles triangle ABC points D and E are taken such that AD is equal to AE, if BE and CD intersect at F, shew that the triangles BFC, DFE are isosceles.

*32. The three straight lines bisecting the sides of a triangle at right angles meet in a point which is equidistant from the vertices of the triangle.

*33. The bisectors of the angles of a triangle meet in a point which is equidistant from the sides of the triangle.

*34. The bisectors of an angle of a triangle and of the exterior angles adjacent to the other two angles meet in a point which is equidistant from the sides of the triangle.

*35. The bisectors of the angles of the triangle ABC meet in O, and OF is drawn perpendicular to AB: shew that AF is equal to the difference between the semi-perimeter of the triangle and the side BC. If OD is perpendicular to BC, find similar values of BD and CD.

*36. The bisector of the angle A of the triangle ABC and of the exterior angles at B and C meet in O, and OD, OE, OF are drawn perpendicular to BC, CA and AB produced when necessary: shew that AE and AF are each equal to the semi-perimeter of the triangle. Find values of BD and CD in terms of the semi-perimeter and the sides.

37. Prove that the perimeter of a triangle is greater than the sum of the straight lines drawn from the vertices to the middle points of the opposite sides.

38. If ABC is a triangle having the side AB less than the

side AC, and the bisector of the angle BAC meet BC at D: shew that BD is less than CD.

39. If ABC is a triangle having the side AB less than the side AC: shew that the bisector of the angle BAC lies between AB and the straight line drawn from A to the middle point of BC.

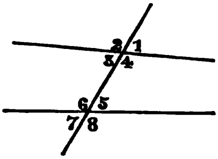
*40. Two points, A and B, lie on the same side of the straight line CD; P is a point in CD, such that AP and BP make equal angles with CD; Q is any other point in CD: shew that the sum of AP and BP is less than the sum of AQ and BQ.

SECTION III.

PARALLELS AND PARALLELOGRAMS.

DEF. 33. **Parallel** *straight lines are such as are in the same plane and being produced to any length both ways do not meet.*

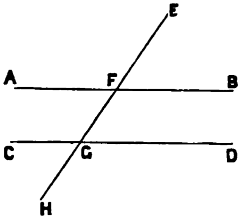
DEF. 34. *When a straight line intersects two other straight lines it makes with them eight angles, which have received special names in relation to the lines or to one another.*



Thus in the figure 1, 2, 7, 8 *are called* **exterior** *angles, and* 3, 4, 5, 6 **interior** *angles; again* 4 *and* 6, 3 *and* 5, *are called* **alternate** *angles; lastly,* 1 *and* 5, 2 *and* 6, 3 *and* 7, 4 *and* 8, *are called* **corresponding** *angles.*

THEOR. 21. If a straight line intersects two other straight lines and makes the alternate angles equal, the straight lines are parallel.

Let the straight line EH intersect the straight lines AB, CD at F and G so as to make the angle AFG equal to the alternate angle FGD:



then shall AB and CD be parallel.

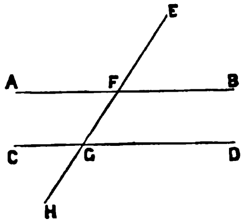
For if AB and CD are not parallel,
they will meet if produced far enough either towards B and D or towards A and C.
Suppose them to meet at a point K;
then, of the two angles AFG and FGD, the one is an exterior, and the other an interior opposite angle, of the triangle FGK;
therefore the angle AFG is not equal to the angle FGD, *I.* 9.
but by hypothesis the angle AFG is equal to the angle FGD,
hence the angles AFG and FGD are both equal and unequal,
which is absurd;

therefore AB and CD cannot meet,
that is, they are parallel. Q.E.D.

Ex. 41. Shew that the contrapositive form of Theor. 21 is included in Theor. 9.

THEOR. 22. If a straight line intersects two other straight lines and makes either a pair of alternate angles equal, or a pair of corresponding angles equal, or a pair of interior angles on the same side supplementary; then, in each case, the two pairs of alternate angles are equal, and the four pairs of corresponding angles are equal, and the two pairs of interior angles on the same side are supplementary.

Let the straight line EFGH intersect the straight lines AB, CD and make the alternate angles AFG, FGD equal:



then shall the alternate angles BFG, FGC be equal, the corresponding angles EFB, FGD equal, and the interior angles BFG, FGD supplementary.

The angle BFG is the supplement of the angle AFG, *I.* 2.
the angle FGC is the supplement of the angle FGD, *I.* 2.
and the angle AFG is equal to the angle FGD, *Hyp.*
therefore the angle BFG is equal to the angle FGC. *I.* 1. *Cor.* 3.

Again, the angle EFB is equal to the vertically opposite angle AFG, *I.* 4.

and the angle FGD is also equal to the angle AFG, *Hyp.*
therefore the angle EFB is equal to the angle FGD. *Ax. c.*

Thirdly, the angle BFG is the supplement of the angle AFG, *I.* 2.
and the angle AFG is equal to the angle FGD, *Hyp.*
therefore the angle BFG is the supplement of the angle FGD.

In like manner the other parts of the Theorem may be demonstrated.

Q.E.D.

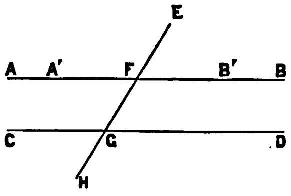
COR. If a straight line intersects two other straight lines and makes a pair of corresponding angles equal, or a pair of interior angles on the same side supplementary, the straight lines are parallel. *By I.* 21.

**Ex.* 42. Straight lines that are perpendicular to the same straight line are parallel to one another.

AXIOM 3. Through the same point there cannot be more than one straight line parallel to a given straight line.

THEOR. 23. If two straight lines are parallel, and are intersected by a third straight line, the alternate angles are equal.

Let the parallel straight lines AB, CD be intersected by the straight line EFGH :



then shall the angle AFG be equal to the alternate angle FGD, and the angle BFG equal to the alternate angle FGC.

Let A′B′ be a straight line through F such that the angle A′FG is equal to the alternate angle FGD,
then A′B′ is parallel to CD, *I.* 21.
but AB is also parallel to CD, *Hyp.*
and there is but one parallel to CD through the point F, *Ax.* 3.
therefore A′B′ must fall along AB,
and therefore the angles AFG and A′FG are the same angle,
therefore the angle AFG is equal to the alternate angle FGD.

In like manner it may be shewn that the angle BFG is equal to the alternate angle FGC.

Q.E.D.

COR. 1. If a straight line intersects two parallel straight lines, and is perpendicular to one of them, it is also perpendicular to the other.

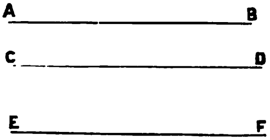
COR. 2. If a straight line intersects two parallel straight lines, it makes the corresponding angles equal, and the interior angles on the same side supplementary. *By I.* 21.

COR. 3. If a straight line falling on two other straight lines makes the interior angles on the same side together less than two right angles, the two straight lines will meet, if continually produced, on the side on which are the angles which are less than two right angles.

**Ex.* 43. If two straight lines are respectively parallel to two other straight lines, the angles made by the first pair are respectively equal to the angles made by the second pair.

THEOR. 24. Straight lines that are parallel to the same straight line are parallel to one another.

Let the straight lines AB, CD be each parallel to the straight line EF :



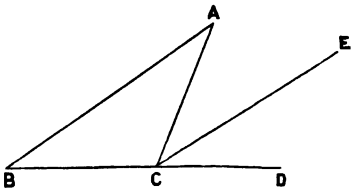
then shall AB and CD be parallel.

For if AB and CD were not parallel, they would meet in some point if produced far enough,
and then through this point there would be two straight lines parallel to the same straight line,
which is impossible, *Ax.* 3.
therefore AB and CD do not meet however far produced,
therefore AB and CD are parallel.

Q.E.D.

THEOR. 25. If a side of a triangle is produced, the exterior angle is equal to the two interior opposite angles; and the three interior angles of a triangle are together equal to two right angles.

Let the side BC of the triangle ABC be produced to D :



then shall the exterior angle ACD be equal to the two interior opposite angles CAB, ABC, and the three interior angles CAB, ABC, BCA together equal to two right angles.

Let the straight line CE make the angle ACE on the side of AC remote from B equal to the angle CAB;
the angle ACD is greater than the interior opposite angle CAB, *I.* 9.
therefore the angle ACD is also greater than the angle ACE,
therefore CE falls within the angle ACD.

Again, because the angle ACE is equal to the alternate angle CAB,
therefore CE and AB are parallel, *I.* 21.
and therefore the angle ECD is equal to the corresponding angle ABC; *I.* 23, *Cor.* 2.
therefore the whole angle ACD is equal to the two angles CAB, ABC. *Ax. d.*

Again, the angles CAB, ABC are together equal to the angle ACD,
to each of these equals add the angle BCA,
then the three angles CAB, ABC, BCA are together equal to the two angles ACD, BCA; *Ax. d.*
but the two angles ACD, BCA are together equal to two right angles, *I.* 2.
therefore the three angles CAB, ABC, BCA are together equal to two right angles.

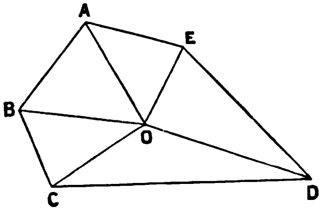
Q.E.D.

COR. In a right-angled triangle the two acute angles are complementary.

**Ex.* 44. If two straight lines are respectively perpendicular to two other straight lines, the angles formed by the first pair are respectively equal to the angles formed by the second pair.

THEOR. 26. All the interior angles of any convex polygon together with four right angles are equal to twice as many right angles as the polygon has sides.

Let ABCDE be any convex polygon :



then shall all the interior angles of ABCDE together with four right angles be equal to twice as many right angles as the polygon has sides.

Take any point O within the polygon ABCDE,
and join O to each of the angular points of the polygon.

Then the polygon is divided into as many triangles OAB, OBC, &c., as the polygon has sides,
and in each triangle the angles are together equal to two right angles, *I.* 25.
therefore all the angles of all the triangles are together equal to twice as many right angles as the polygon has sides.

But the angles at O, the common vertex of all the triangles, make up four right angles, *I.* 2, *Cor.*
and the other angles of the triangles make up the interior angles of the polygon,
therefore the interior angles of the polygon together with four right angles are equal to twice as many right angles as the polygon has sides. Q.E.D.

Ex. 45. Prove Theor. 26 by joining one angular point of the polygon to all the rest.

Ex. 46. Shew that each angle of a regular hexagon is equal to a third part of four right angles.

Ex. 47. What is the magnitude of each angle of a regular pentagon?

Ex. 48. Find the magnitude of each interior angle of an equiangular polygon of n sides.

Ex. 49. A floor is to be covered with tiles, which are identically equal regular figures; what regular figures must they be?

THEOR. 27. The exterior angles of any convex polygon made by producing the sides in order are together equal to four right angles.

For, if each side of the polygon be produced,
the interior angle and exterior angle at each angular point are together equal to two right angles, *I.* 2.
therefore all the interior and all the exterior angles are together equal to twice as many right angles as the polygon has sides;
but the interior angles and four right angles are together equal to twice as many right angles as the polygon has sides, *I.* 26.
therefore the exterior angles are equal to four right angles.
Q.E.D.

Ex. 50. Prove Theor. 27, without using Theor. 26, by using Ex. 43 and Theor. 2, Cor.

Ex. 51. Deduce Theor. 26 from Theor. 27.

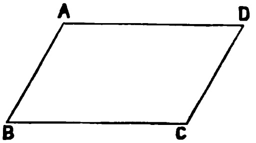
Ex. 52. Each of the exterior angles of a regular polygon is equal to the angle of an equilateral triangle: find the number of sides of the polygon.

DEF. 35. *A* **parallelogram** *is a quadrilateral whose opposite sides are parallel.*

DEF. 36. *A* **trapezium** (*or* **trapezoid**) *is a quadrilateral that has only one pair of opposite sides parallel.*

THEOR. 28. The adjoining angles of a parallelogram are supplementary, and the opposite angles are equal.

Let ABCD be a parallelogram :



then shall any two adjoining angles, as ABC, BCD be supplementary,

and any two opposite angles as ABC, CDA equal.

Because the straight line BC meets the parallel straight lines BA, CD,

therefore the interior angles ABC, BCD are supplementary.

I. 23, *Cor.* 2.

Again, because the angle ABC is the supplement of the adjoining angle BCD,

and the angle CDA is also the supplement of the adjoining angle BCD,
therefore the angle ABC is equal to the angle CDA.

Q.E.D.

COR. If one of the angles of a parallelogram is a right angle, all its angles are right angles.

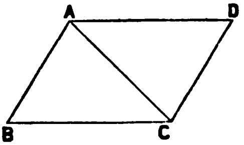
DEF. 37. *A parallelogram, one of whose angles is a right angle, is called a* **rectangle.**

Ex. 53. If the adjoining angles of a quadrilateral are supplementary, the quadrilateral is a parallelogram.

Ex. 54. If the opposite angles of a quadrilateral are equal, the quadrilateral is a parallelogram.

THEOR. 29. The opposite sides of a parallelogram are equal to one another, and each diagonal divides it into two identically equal triangles.

Let ABCD be a parallelogram, having the diagonal AC :



then shall the side AB be equal to the side CD, the side BC to the side DA, and the triangle ABC identically equal to the triangle CDA.

Because AC meets the parallel straight lines AB, DC, therefore the angle BAC is equal to the alternate angle ACD ;

I. 23.

Again, because AC meets the parallel straight lines BC, AD, therefore the angle BCA is equal to the alternate angle CAD; *I.* 23.

and the side AC is common to the two triangles ABC, CDA, therefore the triangles are identically equal, *I.* 6.

the side AB equal to the side CD, and the side BC to the side DA.

Q.E.D.

COR. If the adjoining sides of a parallelogram are equal, all its sides are equal.

DEF. 38. *A* **rhombus** *is a parallelogram that has all its sides equal.*

DEF. 39. *A* **square** *is a rectangle that has all its sides equal.*

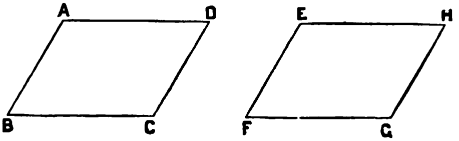
Ex. 55. If the opposite sides of a quadrilateral are equal the quadrilateral is a parallelogram.

**Ex.* 56. The diagonals of a parallelogram bisect one another.

**Ex.* 57. If the diagonals of a quadrilateral bisect one another the quadrilateral is a parallelogram.

THEOR. 30. If two parallelograms have two adjoining sides of the one respectively equal to two adjoining sides of the other, and likewise an angle of the one equal to an angle of the other; the parallelograms are identically equal.

Let ABCD, EFGH be two parallelograms having the angle ABC equal to the angle EFG, and two adjoining sides of the parallelogram ABCD equal to two adjoining sides of the parallelogram EFGH:



then shall the parallelogram ABCD be identically equal to the parallelogram EFGH.

Because the opposite sides of a parallelogram are equal, *I.* 29.
therefore any two adjoining sides are equal to any other two adjoining sides, each to each,
hence the two adjoining sides that contain the angle ABC of the parallelogram ABCD are equal to the two adjoining sides that contain the angle EFG of the parallelogram EFGH, each to each.

Apply the parallelogram ABCD to the parallelogram EFGH, so that B may fall on F, and BC along the side which is equal to BC,
say FG,
and the point A on the same side of FG as the point E;
then C will fall on G,
since BC is equal to FG; *Hyp.*
and BA will fall along FE,
since the angle ABC is equal to the angle EFG; *Hyp.*
and A will fall on E,
since BA is equal to FE. *Hyp.*

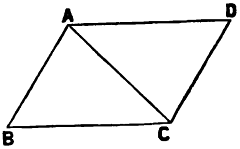
And, A coinciding with E,
AD will fall along EH,
since they are each parallel to FG; *Ax.* 3.
and, C coinciding with G,
CD will fall along GH,
since they are each parallel to FE. *Ax.* 3.
Therefore the point D will coincide with the point H,
and the parallelograms will coincide and be identically equal.

Q.E.D.

COR. **Two rectangles are equal, if two adjoining sides of the one are respectively equal to two adjoining sides of the other; and two squares are equal, if a side of the one is equal to a side of the other.**

THEOR. 31. If a quadrilateral has two opposite sides equal and parallel, it is a parallelogram.

Let the quadrilateral ABCD have the opposite sides AD and BC equal and parallel:



then shall ABCD be a parallelogram.

Join AC.

Then because AC meets the parallel straight lines AD, BC, therefore the angle CAD is equal to the alternate angle ACB. *I.* 23.

Hence, in the triangles CAD, ACB,
the side AD is equal to the side CB, *Hyp.*
the side AC is common to both,
and the angle CAD is equal to the angle ACB,
therefore the angle ACD is equal to the angle CAB, *I.* 5.
and these are alternate angles which AC makes with AB and DC,
therefore AB is parallel to DC, *I.* 21.
therefore ABCD is a parallelogram.

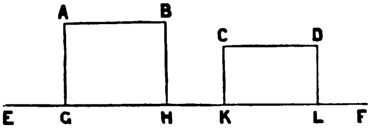
Q.E.D.

DEF. 40. *The* **orthogonal projection of one straight line on another straight line** *is the portion of the latter intercepted between perpendiculars let fall on it from the extremities of the former.*

THEOR. 32. Straight lines that are equal and parallel have equal projections on any other straight line; conversely, parallel straight lines that have equal projections on another straight line are equal.

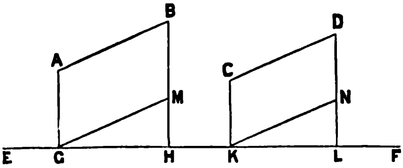
Let AB, CD be two parallel straight lines, GH, KL their projections on any other straight line EF:
then, if AB is equal to CD,
GH shall be equal to KL;
and conversely, if GH is equal to KL,
AB shall be equal to CD.

If AB and CD are parallel to EF,



then AGHB and CKLD are parallelograms,
therefore AB is equal to its projection GH, *I.* 29.
and CD is equal to its projection KL, *I.* 29.
therefore, if AB is equal to CD,
then GH is equal to KL;
and conversely, if GH is equal to KL,
then AB is equal to CD.

But if AB and CD are not parallel to EF,



draw GM parallel to AB, and KN to CD,
meeting BH and DL produced if necessary, at M and N respectively.

Then because GM is parallel to AB, KN to CD, and AB to CD,
therefore GM is parallel to KN, *I.* 24.
therefore the angle MGH is equal to the angle NKL. *I.* 23, *Cor.* 2.

Hence in the triangles MGH, NKL,
the angle MGH is equal to the angle NKL,
and the angle GHM is equal to the angle KLN, each being a right angle;
therefore if AB is equal to CD,
so that GM is equal to KN, *I.* 29.
GH is equal to KL; *I.* 19.
and conversely if GH is equal to KL,
then GM is equal to KN, *I.* 6.
and therefore AB is equal to CD. *I.* 29.

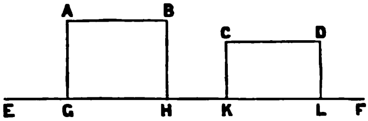
Q.E.D.

THEOR. 33. Equal straight lines that have equal projections on another straight line are parallel to that line, or make equal angles with it.

Let AB, CD be two equal straight lines, GH, KL their projections on another straight line EF, and let GH be equal to KL:

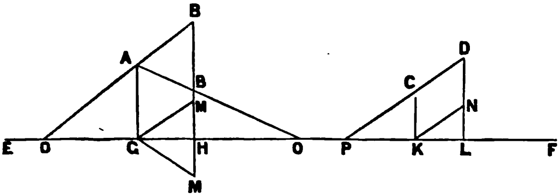
then shall AB and CD be parallel to EF, or make equal angles with EF.

If either AB or CD is parallel to EF, let AB be parallel to EF,



then AGHB is a parallelogram,
and therefore AB is equal to its projection GH, *I.* 29.
therefore CD is also equal to its projection KL; *Hyp.*
but the perpendicular from C upon DL is equal to KL, *I.* 29.
therefore CD is this perpendicular, *I.* 15.
and therefore CD is parallel to EF.
Therefore AB and CD are both parallel to EF if either of them is.

But if neither AB nor CD is parallel to EF,



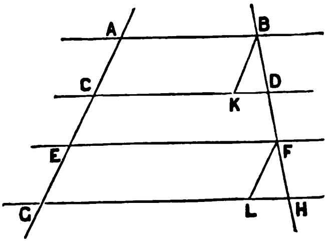
draw GM parallel to AB, and KN to CD, meeting BH and DL produced if necessary at M and N respectively,
let AB, CD produced if necessary meet EF at O and P respectively.

Then since AB is equal to CD *Hyp.*
therefore GM is equal to KN. *I.* 29.
Hence in the right-angled triangles MGH, NKL,
the side GM is equal to the side KN,
and the side GH is equal to the side KL, *Hyp.*
therefore the angle MGH is equal to the angle NKL, *I.* 20, *Cor.* 1.
therefore the angle AOG is equal to the angle CPK, *I.* 23, *Cor.* 2.
that is, AB and CD make equal angles with EF.

Q.E.D.

THEOR. 34. If there are two pairs of straight lines all of which are parallel, and the intercepts made by each pair on a straight line that cuts them are equal, then the intercepts on any other straight line that cuts them are also equal.

Let AB, CD and EF, GH be two pairs of straight lines all of which are parallel, and let the intercepts AC, EG on the straight line AG be equal:



then shall the intercepts BD, FH on any other straight line BH which cuts them be equal.

If BH is parallel to AG,
then BD is equal to AC, *I.* 29.

and FH is equal to EG, *I.* 29.
therefore BD is equal to FH.

But if BH is not parallel to AG,
draw BK and FL parallel to AG, meeting CD and FH in K and L respectively,
then BK is equal to AC, *I.* 29.
and FL is equal to EG, *I.* 29.
therefore BK is equal to FL.
Hence in the triangles BKD, FLH
the angle KBD is equal to the angle LFH, *I.* 23, *Cor.* 2.
the angle BDK is equal to the angle FHL, *I.* 23, *Cor.* 2.
and the side BK is equal to the side FL,
therefore the side BD is equal to the side FH. *I.* 19.

Q.E.D.

COR. 1. If there are three parallel straight lines, and the intercepts made by them on any straight line that cuts them are equal, then the intercepts on any other straight line that cuts them are also equal.

COR. 2. The straight line drawn through the middle point of one of the sides of a triangle parallel to the base passes through the middle point of the other side.

COR. 3. The straight line joining the middle points of two sides of a triangle is parallel to the base.

Ex. 58. The straight lines joining the middle points of the sides of a triangle divide it into four identically equal triangles.

**Ex.* 59. The straight line joining the middle points of two sides of a triangle is equal to half the base.

Ex. 60. If a quadrilateral be formed by joining the middle points of the sides of a given quadrilateral, it is a parallelogram.

EXERCISES.

61. Prove that the angle between the bisectors of two adjoining angles of a quadrilateral is half the sum of the two remaining angles.

62. If the sides of a regular pentagon be produced to meet, the angles formed by these lines are together equal to two right angles.

63. If a straight line parallel to BC, the base of an isosceles triangle ABC, meet the sides AB, AC at D and E: shew that the triangles CDE, DCB have two sides and one angle of the one equal to two sides and one angle of the other. Are they equal in all respects?

64. Straight lines AD, BE, CF are drawn within the triangle ABC, making the angles DAB, EBC, FCA all equal to one another. If the lines AD, BE, CF do not meet at a point: prove that the angles of the triangle formed by them are equal to those of the triangle ABC, each to each.

65. The exterior angles at B and C of the triangle ABC are bisected by lines meeting at D. Shew that the angle BDC is equal to half the exterior angle at A.

66. If a quadrilateral has two sides parallel, and the other two sides equal but not parallel, shew that the diagonals of the quadrilateral are equal.

67. Two triangles ABC, DBC are upon the same base BC, and AD is parallel to BC. If ABC is isosceles, shew that its perimeter is less than that of DBC.

68. The sum of the distances of any point in the base of an isosceles triangle from the two sides is constant.

69. If the base of an isosceles triangle be produced to any point, the difference of the distances of the point from the two sides is constant.

70. Shew that the sum of the distances of any point within an equilateral triangle from the three sides is constant.

71. ABCD is a parallelogram, AD is bisected at E, and BC at F: shew that BE and DF trisect the diagonal AC.

*72. In the triangle ABC, D and E are the middle points of BC and CA: shew that BE cuts off a third part from DA.

*73. Shew that the three straight lines drawn from the vertices of a triangle to the middle points of the opposite sides meet in a point.

SECTION IV.

PROBLEMS.

A Geometrical Problem is a proposition, of which the object is to effect some Geometrical construction.

The solution of a Problem depends on the instruments, the use of which is allowed; and it will be readily understood that the more restricted the choice of instruments, the more limited will be the Problems which can be solved by their use, and the more difficult will the solution of many that are possible be found.

Owing to the existence of this arbitrary element in the treatment of Problems, they are grouped together in a separate section. Though important as applications of Geometrical truths, it should be clearly understood that Problems form no part of the chain of connected truths embodied in the Theorems of Geometry, so that, though they may advantageously be studied in connection with the Theorems on which they directly depend, they are not a necessary part of the pure Science of Geometry.

It is the recognised convention of Elementary Geometry that the only instruments to be employed are—the ruler, for drawing and producing straight lines, and the compasses for describing circles and for the transference* of distances.

This Convention is embodied in the following

POSTULATES OF CONSTRUCTION.

Let it be granted that

1. **A straight line may be drawn from any one point to any other point.**
2. **A terminated straight line may be produced to any length in a straight line.**
3. **A circle may be drawn with any centre, with a radius equal to any finite straight line.**

DEF. 41. *A* **circle** *is a plane figure contained by one line, which is called the* **circumference**, *and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another. This point is called the* **centre** *of the circle.*

DEF. 42. *A* **radius** *of a circle is a straight line drawn from the centre to the circumference.*

DEF. 43. *A* **diameter** *of a circle is a straight line drawn through the centre and terminated both ways by the circumference.*

It follows from the definition of a circle that a point is within a circle, when its distance from the centre is less than the radius of the circle.

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* NOTE.—Euclid restricts the use of the compasses to describing circles, and shews in his Props. 2 and 3 how with this restriction to draw a line of given length and to cut off a given length from a given line.

It is evident that

(1) If a point in a straight line is within a closed figure the straight line if produced in either sense from the point will meet the boundary of the figure, and thus intersect it in two points at least;

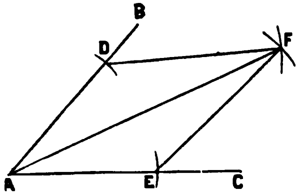
(2) If a point in the boundary of one closed figure lie within another closed figure, and also a point in the boundary of the latter lie within the former, the two boundaries intersect in two points at least;

For they cannot lie wholly outside each of the other: and if one were wholly inside the other, no point in the boundary of the second would lie within the first. Hence they must lie partly inside and partly outside each of the other, and their boundaries (circumferences) must meet in two points at least.

By the help of the above, it may be shewn that the straight lines and circles drawn in the Problems of this section intersect; or the conditions that must be satisfied in order that they may intersect may be determined.

PROB. 1. To bisect a given angle.

Let BAC be the given angle: it is required to bisect it.



With centre A, and with any radius, draw a circle cutting AB at D, and AC at E. *Post.* 3.
With centres D and E, and with any radius greater than half the straight line DE, draw circles, *Post.* 3.
let F be a point of intersection of these circles which lies within the angle BAC;
join AF: *Post.* 1.
then shall AF bisect the angle BAC.

Join DF, EF.
Then in the triangles DAF, EAF,
the side AD is equal to the side AE, *Constr.*
the side AF is common to both,
and the side DF is equal to the side EF, *Constr.*
therefore the angle DAF is equal to the angle EAF, *I.* 18.
therefore AF bisects the angle BAC.

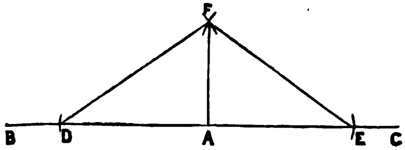
Q.E.F.

Ex. 74. Divide a given angle into four equal parts.

Ex. 75. Prove that FA produced will bisect the major conjugate angle BAC.

PROB. 2. To draw a perpendicular to a given straight line from a given point in it.

Let BAC be the given straight line, A the given point in it: it is required to draw from A a perpendicular to BAC.



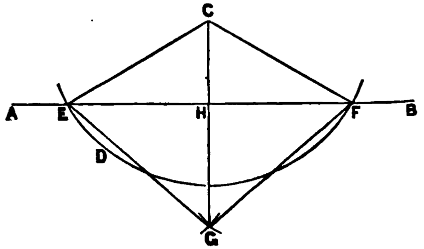
With centre A, and with any radius, draw a circle cutting AB at D, and AC at E, *Post.* 3.
With centres D and E, and with any radius greater than AD or AE, draw circles, *Post.* 3.
let F be a point of intersection of these circles;
join AF: *Post.* 1.
then shall AF be perpendicular to BAC.

Join DF, EF.
Then in the triangles DAF, EAF,
the side AD is equal to the side AE, *Constr.*
the side AF is common to both,
and the side DF is equal to the side EF, *Constr.*
therefore the angle DAF is equal to the angle EAF, *I.* 18.
therefore AF is perpendicular to BAC. *Defs.* 10, 11.
Q.E.F.

Ex. 76. Shew that Prob. 2 is a particular case of Prob. 1.

PROB. 3. To draw a perpendicular to a given straight line from a given point outside it.

Let AB be the given straight line, C the given point outside it:
it is required to draw a perpendicular to AB from C.



Take any point D on the side of AB remote from C,
with centre C and radius CD draw a circle cutting AB produced if necessary at E and F, *Post.* 3.
with centres E and F and the same radius as before draw circles cutting one another, on the side of AB remote from C, at G, *Post.* 3.
join CG cutting AB at H:
then shall CH be perpendicular to AB.

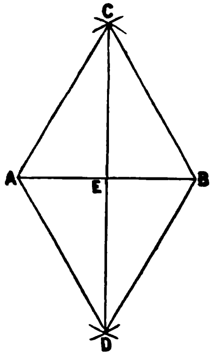
Join CE, CF, EG, FG. *Post.* 1.
Then in the triangles ECG, FCG,
the side CE is equal to the side CF, *Constr.*
the side EG is equal to the side FG, *Constr.*
and the side CG is common to both,
therefore the angle ECG is equal to the angle FCG. *I.* 18.
Again, in the triangles ECH, FCH,
the side EC is equal to the side FC, *Constr.*
the side CH is common to both,
and the angle ECH is equal to the angle FCH,
therefore the angle CHE is equal to the angle CHF, *I.* 5.
therefore CH is perpendicular to AB. *Def.* 10.
Q.E.F.

Ex. 77. Why is the point D taken on the side of AB remote from C?

Ex. 78. Shew, by Theor. 15, Cor., that the circle whose centre is C cannot meet AB in more than two points.

PROB. 4. To bisect a given finite straight line.

Let AB be the given finite straight line
it is required to bisect it.



With centres A and B and radii equal to AB draw circles cutting one another at C and D, *Post.* 3.
join CD, cutting AB at E : *Post.* 1.
then shall AB be bisected at E.

Join AC, AD, BC, BD. *Post.* 1.

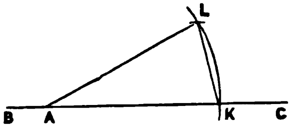
Then in the triangles ACD, BCD,
the side AC is equal to the side BC, *Constr.*
the side AD is equal to the side BD, *Constr.*
and the side CD is common to both,
therefore the angle ACD is equal to the angle BCD. *I.* 18.

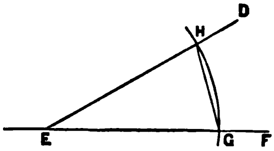
Again, in the triangles ACE, BCE,
the side AC is equal to the side BC, *Constr.*
the side CE is common to both,
and the angle ACE is equal to the angle BCE,
therefore the side AE is equal to the side BE, *I.* 5.
therefore AB is bisected at E.

Q.E.F.

PROB. 5. At a given point in a given straight line to make an angle equal to a given angle.

Let A be the given point in the given straight line BC, DEF the given angle:
it is required to draw from A a straight line making with BC an angle equal to DEF.





With centre E and any radius draw a circle cutting EF and ED at G and H. *Post.* 3.
Join GH,
with centre A and the same radius draw a circle cutting BC at K, *Post.* 3.
with centre K and radius equal to GH draw a circle cutting the last circle at L, *Post.* 3.
join AL:
then shall AL make with BC an angle equal to DEF.

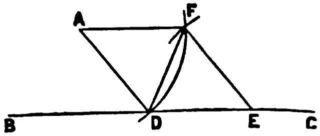
Join KL.

Then in the triangles LAK, HEG,
the side AK is equal to the side EG, *Constr.*
the side AL is equal to the side EH, *Constr.*
and the side LK is equal to the side HG, *Constr.*
therefore the angle LAK is equal to the angle HEG, *I.* 18.
that is, AL makes with BC an angle equal to the angle DEF.

Q.E.F.

PROB. 6. To draw a straight line through a given point parallel to a given straight line.

Let A be the given point, BC the given straight line:
it is required to draw through A a straight line parallel to BC.



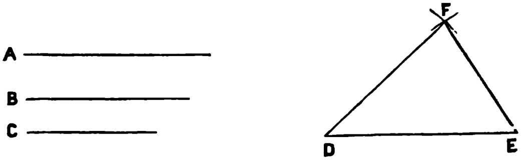
In BC take any point D, join AD;
with centre A and radius AD draw a circle; *Post.* 3.
from DC cut off DE equal to AD; *Post.* 3.
with centre E and radius ED draw a circle cutting the former circle at F; *Post.* 3.
join AF: *Post.* 1.
then shall AF be parallel to BC.

Join DF, EF.
Then, in the triangles ADF, EFD,
the side AF is equal to the side ED, *Constr.*
the side FD is common to both,
and the side AD is equal to the side EF, *Constr.*
therefore the angle AFD is equal to the angle EDF, *I.* 18.
and these are alternate angles made by DF with AF and BC,
therefore AF is parallel to BC. *I.* 21.
Q.E.F.

Ex. 79. Through a given point draw a straight line so that the part intercepted between two given parallel straight lines may be equal to a given straight line.

PROB. 7. To construct a triangle when three sides are given.

Let A, B and C be the three given sides:
it is required to construct a triangle having its sides equal to A, B and C.



Let A be not less than either B or C.
Take any straight line DE equal to A, *Post.* 3.
With centre D, and radius equal to B, draw a circle, *Post.* 3.
with centre E, and radius equal to C, draw a circle cutting the former circle at F, *Post.* 3.
join DF and EF: *Post.* 1.
then shall FDE be the triangle required.

For DE is equal to A, *Constr.*
DF is equal to B, *Constr.*
and EF is equal to C, *Constr.*
therefore FDE is the triangle required.

Q.E.F.

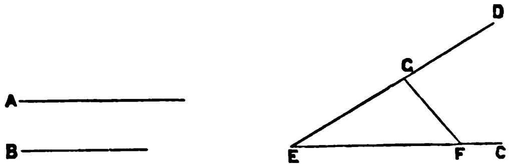
Ex. 80. Point out how the construction fails if one side is greater than the sum of the other two.

**Ex.* 81. Construct an equilateral triangle on a given base.

**Ex.* 82. Trisect a right angle.

PROB. 8. To construct a triangle when two sides and the angle between them are given.

Let A, B be the two given sides, CED the given angle: it is required to construct a triangle having two of its sides equal to A and B, and the angle between them equal to CED.



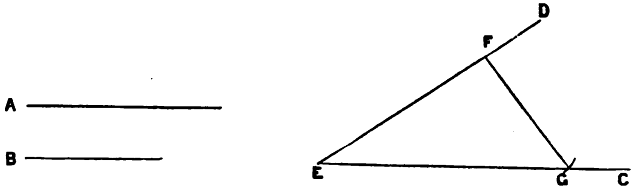
From EC cut off EF equal to A, *Post.* 3.
from ED cut off EG equal to B, *Post.* 3.
join FG: *Post.* 1.
then shall FEG be the triangle required.

For EF is equal to A, *Constr.*
EG is equal to B, *Constr.*
and the angle FEG is the angle CED,
therefore FEG is the triangle required.

Q.E.F.

PROB. 9. To construct a triangle when two sides and an angle opposite to one of them are given.

Let A, B be the two given sides, CED the given angle opposite to the side equal to B:
it is required to construct a triangle having two of its sides equal to A and B, and the angle opposite to B equal to CED.



From ED cut off EF equal to A, *Post.* 3.
with centre F, and radius equal to B, draw a circle. *Post.* 3.

If B is less than the perpendicular from F upon EC,
the circle does not meet EC, and the solution is impossible.

If B is equal to the perpendicular
the circle meets EC in one point only, and there is one, and only one, solution.

If B is greater than the perpendicular, and less than A,
the circle cuts EC at two points on the same side of the vertex of the given angle, and there are two solutions, or the solution is ambiguous.

If B is greater than A,
the circle cuts EC at two points on opposite sides of the vertex of the given angle, and there is one, and only one, solution.

Let G be a point in which the circle meets EC,
join FG: *Post.* 1.
then shall FEG be the triangle required.

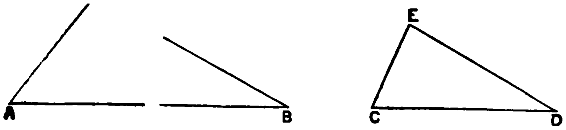
For EF is equal to A, *Constr.*
FG is equal to B, *Constr.*
and the angle FEG is the angle CED,
therefore FEG is the triangle required.

Q.E.F.

Ex. 83. Construct a right-angled triangle having given the hypotenuse and one side.

PROB. 10. To construct a triangle, when two angles and the side between their vertices are given.

Let A, B be the two given angles, CD the given side: it is required to construct a triangle having two of its angles equal to A and B, and the side between their vertices equal to CD.



At the point C, in the straight line CD, make the angle DCE equal to the angle A; *Prob.* 5.
at the point D, in the straight line CD, make the angle CDE equal to the angle B, *Prob.* 5.
then shall CDE be the triangle required.

For the angle DCE is equal to A, *Constr.*
and the angle CDE is equal to B, *Constr.*
and CD is a side of the triangle,
therefore CDE is the triangle required.

Q.E.F.

Ex. 84. Shew that the solution is possible only when the two given angles are together less than two right angles.

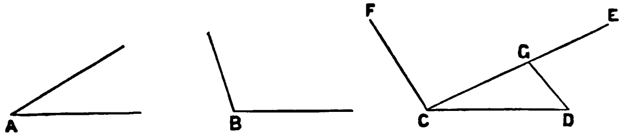
Ex. 85. Construct an isosceles right-angled triangle on a given straight line as hypotenuse.

Ex. 86. On a given straight line as hypotenuse construct a right-angled triangle having one of its acute angles double the other: shew that the hypotenuse is double the shorter side.

Ex. 87. Draw a straight line parallel to the base of a given triangle, so that the portion intercepted between the sides may be equal to a given straight line.

PROB. 11. To construct a triangle, when two angles and a side opposite to one of them are given.

Let A, B be the given angles, CD the given side opposite to the angle B:
it is required to construct a triangle having two of its angles equal to A and B, and the side opposite to the angle equal to B equal to CD.



At the point C, in the straight line CD, make the angle DCE equal to the angle A; *Prob.* 5.
at the point C, in the straight line CE, make the angle ECF equal to the angle B; *Prob.* 5.
through the point D draw DG parallel to CF meeting CE at G: *Prob.* 6.
GCD shall be the triangle required.

For, because DG is parallel to CF,
therefore the angle DGC is equal to the alternate angle GCF, *I.* 23.

that is, to the angle B ; *Constr.*
the angle DCG is equal to the angle A ; *Constr.*
and CD is a side of the triangle ;
therefore GCD is the triangle required.

Q.E.F.

EXERCISES.

89. Trisect a given straight line.
90. Through a given point O within given angle BAC draw a straight line BOC, such that BC may be bisected at O.
91. Through a given point O without a given angle BAC draw a straight line OBC, such that OB may be equal to BC.
92. Through a given point O within a given angle BAC draw a straight line BOC, such that BO may be double OC.
93. Through a given point O without a given angle BAC draw a straight line OBC, such that OB may be double BC.
94. Given a point, a closed curve which incloses the point, and a straight line outside the curve, draw a straight line terminated by the point and the straight line, and bisected by the curve.
95. Find a point in a given straight line, such that straight lines drawn from it to two given points on the same side of the given straight line may be equally inclined to the given straight line.
96. Construct a triangle having given the base, an angle djacent to the base, and the sum of the remaining sides.
97. Construct a triangle having given the base, an adjacent angle, and the difference of the remaining sides.
98. Construct a triangle having given the base, the opposite angle, and the sum of the remaining sides.

99. Construct a triangle having given the base, the opposite angle, and the difference of the remaining sides.

100. Construct a triangle having given the base, the difference of the base angles, and the difference of the remaining sides.

101. Construct a triangle having given the perimeter and the angles.

102. Construct a right-angled triangle, having given the hypotenuse and the perpendicular upon it from the right angle.

SECTION V.

PLANE LOCI.

It may happen that the conditions given for the determination of a point do not suffice to fix its position absolutely, but are sufficient to limit it to some line, part of a line, or group lines. In such cases the point is said to have a *locus*.

DEF. 44. *If any and every point on a line, part of a line, or group of lines (straight or curved), satisfies an assigned condition, and no other point does so, then that line, part of a line, or group of lines, is called the* locus *of the point satisfying that condition.*

Hence, in order that a line or group of lines X may be properly termed the locus of a point satisfying an assigned condition A, it is necessary and sufficient to demonstrate the following pair of associated theorems:

If a point satisfies A, it is upon X, (1)

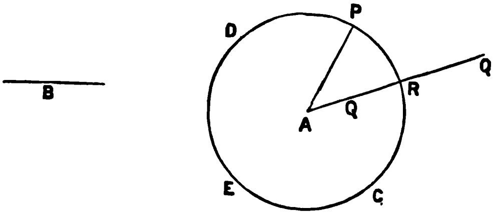
If a point is upon X, it satisfies A. (2)

Instead of (1) we may prove that

If a point is not upon X, it does not satisfy A,
and instead of (2) that
If a point does not satisfy A, it is not upon X.

LOCUS i. To find the locus of a point at a given distance from a given point.

Let A be the given point, B the given distance:
it is required to find the locus of a point at distance B from A.



With centre A, and with radius equal to B, draw the circle CDE:
the circumference CDE shall be the locus required.

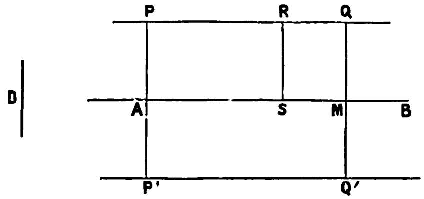
Take any point P on the circumference CDE,
join AP,
then AP is a radius of the circle CDE,
and is therefore equal to B. *Def.* 41.

Take any point Q not in the circumference CDE,
join AQ, and let AQ, produced if necessary, meet the circumference in R,
since Q and R do not coincide, AQ is not equal to AR, *Ax. a.*
but AR is equal to B, *Def.* 41.
therefore AQ is not equal to B.

Hence the circumference CDE is the locus of a point at a distance from A equal to B.

LOCUS ii. To find the locus of a point at a given distance from a given straight line.

Let D be the given distance, and AB the given straight line:
it is required to find the locus of a point at the distance D from AB.



Let Q be a point at the distance D from AB.
Draw AP perpendicular to AB on the same side of AB as Q, and equal to D.
Draw a straight line through P and Q,
and draw QM perpendicular to AB.
Then QM is equal to D, and therefore to PA,
and it is perpendicular to AB,
and therefore parallel to PA. *I. 22 Cor.*
Hence PQ is parallel to AM, *I. 31.*
therefore Q lies upon the straight line drawn through P parallel to AB.

Also every point on this line is at the same distance from AB.

Take any point R upon the line PQ,
and draw RS perpendicular to AB.
Then RS is parallel to PA, *I.* 22, *Cor.*
and RSAP is a parallelogram,
therefore RS is equal to PA, *I.* 29.
therefore RS is equal to D.

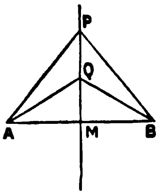
Hence a straight line through P parallel to AB is the locus of a point, on the same side of AB as P, at a distance from AB equal to D.

Similarly, by drawing AP′ in a similar manner on the other side of AB, the line P′Q′ through P′ parallel to AB will be the locus of a point, on the other side of AB, at a distance from AB equal to D.

Hence the locus of a point at a distance from AB equal to D is the pair of parallel straight lines PQ and P′Q′.

LOCUS iii. To find the locus of a point equidistant from two given points.

Let A, B be the two given points :
it is required to find the locus of a point equidistant from A and B.



Let P be any point such that PA is equal to PB,
join AB,
bisect AB in M, *Prob.* 4
and join PM.

Then, in the triangles AMP, BMP
the side AM is equal to the side BM,
the side PM is common to both,
and the side AP is equal to the side BP, *Hyp.*
therefore the angle AMP is equal to the angle BMP, *I.* 18.
therefore they are right angles,
therefore P is on the straight line drawn through M perpendicular to AB.

Also every point on this straight line is equidistant from A and B.

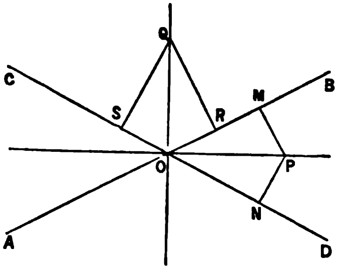
Let Q be any point on MP,
join QA and QB.
Then in the triangles AMQ, BMQ
the side AM is equal to the side BM,
the side MQ is common to both,
and the angle AMQ is equal to the angle BMQ, since they are both right angles,
therefore QA is equal to QB. *I.* 5.

Hence the straight line drawn through M perpendicular to AB is the locus of a point equidistant from A and B.

LOCUS iv. To find the locus of a point equidistant from two intersecting straight lines.

Let AB and CD be two straight lines intersecting at O:

it is required to find the locus of a point equidistant from AB and CD.



Let P be any point such that PM, the perpendicular on AB, is equal to PN, the perpendicular on CD,
join OP.

Then in the right-angled triangles OPM, OPN,
the side PM is equal to the side PN,
and the side OP is common to both,
therefore the angle POM is equal to the angle PON,

I. 20, *Cor.* 1.

therefore P is on one of the bisectors of the angles between AB and CD.

Also every point on these bisectors is equidistant from AB and CD.
Let Q be such a point,
draw QR perpendicular to AB,
and QS perpendicular to CD.
Then in the triangles QRO, QSO,
the angle QRO is equal to the angle QSO,
the angle QOR is equal to the angle QOS,
and the side QO is common to both,
therefore QR is equal to QS. *I.* 19.

Hence the bisectors of the angles between AB and CD are the locus of a point equidistant from AB and CD.

EXERCISES.

103. Find the locus of a point at a given distance from the circumference of a given circle.
104. Find the locus of the vertex of an isosceles triangle on a given base.
105. Find the locus of the middle point of a straight line drawn from a given point to a given unlimited straight line which does not pass through the point.
106. Given the sum (or the difference) of the distances of a point from two intersecting and unlimited straight lines, find its locus.

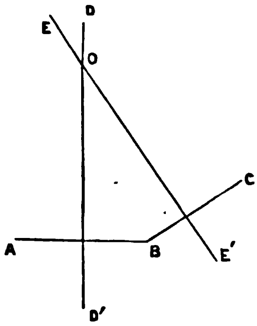
INTERSECTION OF LOCI.

It follows from Def. 44 that if X is the locus of a point satisfying the condition A, and Y the locus of a point satisfying the condition B; then the intersections of X and Y, and these points only, satisfy both the conditions A and B.

i. To find a point equidistant from three given points not in the same straight line.

Let A, B, C be three given points not in the same straight line,

it is required to find a point equidistant from A, B and C.



Draw the straight line DD′ bisecting AB at right angles. *Prob.* 4.

and EE′ bisecting BC at right angles. *Prob.* 4.

Then because every point equidistant from A and B lies on DD′, *Locus iii.*

and every point equidistant from B and C lies on EE′, *Locus iii.*

therefore any point equidistant from A, B and C must lie on DD′ and on EE′.

Now DD′ and EE′ intersect, since if they were parallel, AB and BC, which are perpendicular to them, would lie in one straight line;

let them intersect in O,

then a point O has been found equidistant from A, B and C.

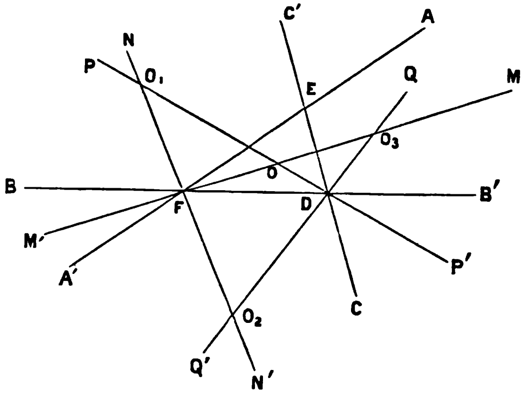
Also, because DD′ and EE′ can intersect in one point only, *Ax.* 2.

therefore O is the only point equidistant from A, B and C.

ii. To find a point equidistant from three given straight lines which intersect one another, but not in the same point.

Let AA′, BB′, CC′ be three given straight lines which intersect so as to form a triangle DEF :

it is required to find a point equidistant from AA′, BB′, and CC′.



Draw the pair of straight lines MM′, NN′ bisecting the angles formed by AA′ and BB′, *Prob.* 1.
and the pair of straight lines PP′, QQ′, bisecting the angles formed by BB′ and CC′. *Prob.* 1.

Then because every point equidistant from AA′ and BB′ lies on MM′ or NN′, *Locus iv.*
and every point equidistant from BB′ and CC′ lies on PP′ or QQ′, *Locus iv.*
therefore any point equidistant from AA′, BB′ and CC′ must lie on MM′ or NN′, and also on PP′ or QQ′.

Now MM′ and PP′ are not parallel,
since the angles MFD, PDF, being half the angles EFD, EDF of the triangle DEF, are not together equal to two right angles;
also MM′ and QQ′ are not parallel,
since the angle MFD, being half the angle EFD, is not equal to

the angle QDB′, which is half the exterior angle EDB′ of the triangle DEF ;
let MM′ intersect PP′ in O, and QQ′ in O_3.
Similarly let NN′ intersect PP′ in O_1, and QQ′ in O_2.
Then four points O, O_1. O_2, O_3 have been found equidistant from AA′, BB′, and CC′.

Also, because two straight lines can intersect in one point only, *Ax.* 2.
therefore O, O_1, O_2, O_3 are the only points equidistant from AA′, BB′, and CC′.

EXERCISES.

107. Find the points in a given straight line which are at a given distance from a given point.
108. In a given straight line find points at a given distance from another given straight line.
109. In a given straight line find a point equidistant from two given points.
110. Three unlimited straight lines form a triangle: find points in one of them which are equidistant from the other two.
111. How many points are there in a plane each of which is equidistant from two given unlimited straight lines, as well as from two given points situated in that plane?

DEFINITIONS, AXIOMS, AND POSTULATES OF BOOK I.

DEFINITIONS.

DEF. 1. *A* **point** *has position, but it has no magnitude.*

DEF. 2. *A* **line** *has position, and it has length, but neither breadth nor thickness.*

The extremities of a line are points, and the intersection of two lines is a point.

DEF. 3. *A* **surface** *has position, and it has length and breadth, but not thickness.*

The boundaries of a surface, and the intersection of two surfaces, are lines.

DEF. 4. *A* **solid** *has position, and it has length, breadth and thickness.*

The boundaries of a solid are surfaces.

DEF. 5. *A* **straight line** *is such that any part will, however placed, lie wholly on any other part, if its extremities are made to fall on that other part.*

DEF. 6. *A* **plane** *surface, or plane, is a surface in which any two points being taken the straight line that joins them lies wholly in that surface.*

DEF. 7. *When two straight lines are drawn from the same point, they are said to contain, or to make with each other, a* **plane angle.** The point is called the *vertex*, and the straight lines are called the *arms*, of the angle.

A line drawn from the vertex and turning about the vertex in the plane of the angle from the position of coincidence with one arm to that of coincidence with the other is said to turn through the angle: and the angle is greater as the quantity of turning is greater.

Since the line may turn from the one position to the other in either of two ways, two angles are formed by two straight lines drawn from a point. These angles (which have a common vertex and common arms) are said to be *conjugate.* The greater of the two is called the *major conjugate*, and the smaller the *minor conjugate*, angle.

When *the angle contained by two lines* is spoken of without qualification, the *minor conjugate* angle is to be understood. It is seldom requisite to consider major conjugate angles before Book III.

When the arms of an angle are in the same straight line, the conjugate angles are equal, and each is then said to be a **straight angle.**

DEF. 8. *When three straight lines are drawn from a point, if one of them be regarded as lying between the other two, the angles which this one (the mean) makes with the other two (the extremes) are said to be* **adjacent** *angles: and the angle between the extremes, through which a line would turn in passing from one extreme through the mean to the other extreme, is the sum of the two adjacent angles.*

DEF. 9. *The* **bisector** *of an angle is the straight line that divides it into two equal angles.*

DEF. 10. *When one straight line stands upon another straight line and makes the adjacent angles equal, each of the angles is called a* **right angle.**

DEF. 11. *A* **perpendicular** *to a straight line is a straight line that makes a right angle with it.*

DEF. 12. *An* **acute** *angle is that which is less than a right angle.*

DEF. 13. *An* **obtuse** *angle is that which is greater than one right angle, but less than two right angles.*

DEF. 14. *A* **reflex** *angle is a term sometimes used for a major conjugate angle.*

DEF. 15. *When the sum of two angles is a right angle, each is called the* **complement** *of the other, or is said to be* **complementary** *to the other.*

DEF. 16. *When the sum of two angles is two right angles, each is called the* **supplement** *of the other, or is said to be* **supplementary** *to the other.*

DEF. 17. *The opposite angles made by two straight lines that intersect are called* **vertically opposite angles.**

DEF. 18. *A plane* **figure** *is a portion of a plane surface inclosed by a line or lines.*

DEF. 19. *Figures that may be made by superposition to coincide with one another are said to be* **identically** *equal; or they are said to be equal in all respects.*

DEF. 20. *The* **area** *of a plane figure is the quantity of the plane surface inclosed by its boundary.*

DEF. 21. *A* **plane rectilineal figure** *is a portion of a plane surface inclosed by straight lines. When there are more than three inclosing straight lines the figure is called a* **polygon.**

DEF. 22. *A polygon is said to be* **convex** *when no one of its angles is reflex.*

DEF. 23. *A polygon is said to be* **regular** *when it is equilateral and equiangular; that is, when its sides and angles are equal.*

DEF. 24. *A* **diagonal** *is the straight line joining the vertices of any angles of a polygon which have not a common arm.*

DEF. 25. *The* **perimeter** *of a rectilineal figure is the sum of its sides.*

DEF. 26. *A* **quadrilateral** *is a polygon of four sides, a* **pentagon** *one of five sides, a* **hexagon** *one of six sides, and so on.*

DEF. 27. *A* **triangle** *is a figure contained by three straight lines.*

DEF. 28. *Any side of a triangle may be called the* **base,** *and the opposite angular point is then called the* **vertex.**

DEF. 29. *An* **isosceles** *triangle is that which has two sides equal; the angle contained by those sides is called the* **vertical** *angle, the third side the* **base.**

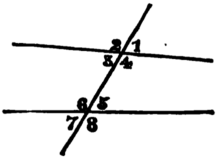
DEF. 30. *A triangle which has one of its angles a right-angle is called a* **right-angled** *triangle. A triangle which has one of its angles an obtuse angle is called an* **obtuse-angled** *triangle. A triangle which has all its angles acute is called an* **acute-angled** *triangle.*

DEF. 31. *The side of a right-angled triangle which is opposite to the right-angle is called the* **hypotenuse.**

DEF. 32. *The perpendicular to a given straight line from a given point outside it is called the* **distance** *of the point from the straight line.*

DEF. 33. **Parallel** *straight lines are such as are in the same plane and being produced to any length both ways do not meet.*

DEF. 34. *When a straight line intersects two other straight lines it makes with them eight angles, which have received special names in relation to the lines or to one another.*



Thus in the figure 1, 2, 7, 8 *are called* **exterior** *angles, and* 3, 4, 5, 6 **interior** *angles; again* 4 *and* 6, 3 *and* 5, *are called* **alternate** *angles; lastly,* 1 *and* 5, 2 *and* 6, 3 *and* 7, 4 *and* 8, *are called* **corresponding** *angles.*

DEF. 35. *A* **parallelogram** *is a quadrilateral whose opposite sides are parallel.*

DEF. 36. *A* **trapezium** (*or* **trapezoid**) *is a quadrilateral that has only one pair of opposite sides parallel.*

DEF. 37. *A parallelogram, one of whose angles is a right angle, is called a* **rectangle.**

DEF. 38. *A* **rhombus** *is a parallelogram that has all its sides equal.*

DEE. 39. *A* **square** *is a rectangle that has all its sides equal.*

DEF. 40. *The* **orthogonal projection of one straight line on another straight line** *is the portion of the latter intercepted between perpendiculars let fall on it from the extremities of the former.*

DEF. 41. *A* **circle** *is a plane figure contained by one line, which is called the* **circumference,** *and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another. This point is called the* **centre** *of the circle.*

DEF. 42. *A* **radius** *of a circle is a straight line drawn from the centre to the circumference.*

DEF. 43. *A* **diameter** *of a circle is a straight line drawn through the centre and terminated both ways by the circumference.*

DEF. 44. *If any and every point on a line, part of a line, or group of lines (straight or curved), satisfies an assigned condition, and no other point does so, then that line, part of a line, or group of lines, is called the* **locus** *of the point satisfying that condition.*

GEOMETRICAL AXIOMS.

AXIOM 1. Magnitudes that can be made to coincide are equal.

AXIOM 2. Through two points there can be made to pass one, and only one, straight line; and this may be indefinitely prolonged either way.

Hence,

α. Any straight line may be made to fall on any other straight line with any given point on the one on any given point on the other;

β. Two straight lines which meet in one point cannot meet again, unless they coincide.

AXIOM 3. Through the same point there cannot be more than one straight line parallel to a given straight line.

POSTULATES OF CONSTRUCTION.

Let it be granted that

1. **A straight line may be drawn from any one point to any other point.**
2. **A terminated straight line may be produced to any length in a straight line.**
3. **A circle may be drawn with any centre, with a radius equal to any finite straight line.**

BOOK II.

EQUALITY OF AREAS.

SECTION I.

THEOREMS.

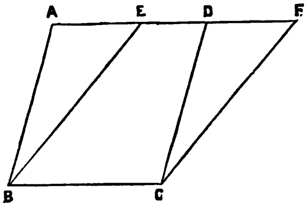
DEF 1. *The* **altitude** *of a parallelogram with reference to a given side as base is the perpendicular distance between the base and the opposite side.*

DEF 2. *The* **altitude** *of a triangle with reference to a given side as base is the perpendicular distance between the base and the opposite vertex.*

It follows from the General Axioms (*d*) and (*e*), as an extension of the Geometrical Axiom 1, that magnitudes which are either the sums or the differences of identically equal magnitudes are equal, although they may not be identically equal.

THEOR. I. Parallelograms on the same base and between the same parallels are equal.

Let ABCD, EBCF be two parallelograms on the same base BC, and between the same parallels AF, BC:



then shall the parallelogram ABCD be equal to the parallelogram EBCF.

In the triangles DCF, ABE,
the angle FDC is equal to the angle EAB,
since DC is parallel to AB, *I.* 23, *Cor.* 2.
the angle CFD is equal to the angle BEA,
since CF is parallel to BE, *I.* 23, *Cor.* 2.
and the side CD is equal to the side BA,
since they are opposite sides of a parallelogram, *I.* 29.
therefore the triangles are identically equal. *I.* 19.

From the quadrilateral ABCF take away the triangle DCF, and from the same quadrilateral take away the equal triangle ABE,
then the remainder ABCD is equal to the remainder EBCF.

Q.E.D.

COR. 1. A parallelogram is equal to a rectangle, whose base and altitude are equal to those of the parallelogram.

For by the Theorem a parallelogram is equal to the rectangle on the same base, and of the same altitude, and by *I.* 30, *Cor.*, this rectangle is equal to any other rectangle whose base and altitude are respectively equal to those of the other.

COR. 2. Parallelograms on equal bases and of equal altitudes are equal; and of parallelograms of equal altitudes, that is the

greater which has the greater base; and also of parallelograms on equal bases, that is the greater which has the greater altitude.

Parallelograms on equal bases and of equal altitudes are equal, since each parallelogram is equal to a rectangle whose base and altitude are equal to those of the parallelogram.

Again, rectangles of equal altitudes but on unequal bases may be superposed so that the rectangle which has the greater base is a whole of which the other rectangle is a part, and there fore less than the whole; hence, because a parallelogram is equal to a rectangle whose base and altitude are equal to those of the parallelogram, it follows that of parallelograms of equal altitudes that is the greater which has the greater base.

In like manner it may be shown that of parallelograms on equal bases, that is the greater which has the greater altitude.

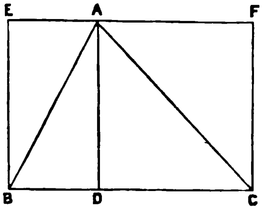
Ex. 1. Prove Theor. 1 by *I.* 5, arranging the proof in three cases for the different positions of AD and EF.

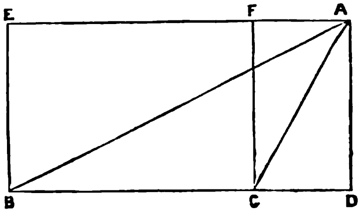
Ex. 2. Prove Theor. 1 in the case in which D and E coincide, by the second part of *I.* 29.

Ex. 3. Shew how to divide any two parallelograms on the same base and between the same parallels into parts which are identically equal each to each.

THEOR. 2. A triangle is equal to half a rectangle whose base and altitude are equal to those of the triangle.

Let ABC be a triangle, and AD its altitude:





then shall the triangle ABC be equal to half a rectangle whose base is equal to BC and altitude to AD.

From B and C draw BE and CF parallel to AD, and let them meet EAF drawn through A parallel to BC in E and F.

Then EC is a rectangle whose base is BC and altitude AD.
Now the triangle ABD is half the rectangle ED, *I.* 29.
and the triangle ADC is half the rectangle AC, *I.* 29.
therefore in fig. 1 the sum, or in fig. 2 the difference of the triangles ABD, ADC is half the sum or difference of the rectangles ED, AC,
therefore the triangle ABC is half the rectangle EC.

Q.E D.

COR. 1. Triangles on the same or equal bases and of equal altitudes are equal.

For each triangle is half a rectangle whose base and altitude are equal to those of the triangles, and such rectangles are equal by *I.* 30, *Cor.*

COR. 2. Equal triangles on the same or equal bases have equal altitudes.

For triangles on the same or equal bases and of unequal altitudes are halves of rectangles on equal bases and of unequal altitudes (Theor. 2), and the rectangle having the greater altitude being greater than the other, so is the triangle of greater altitude greater than the other. Hence, by contraposition, if the triangles are equal their altitudes are also equal.

COR. 3. If two equal triangles stand on the same base and on the same side of it, or on equal bases in the same straight line and on the same side of that straight line, the line joining their vertices is parallel to the base or to that straight line.

For the altitudes of the triangles are parallel since they are both perpendicular to the straight line in which the bases lie, and they are equal by hypothesis, therefore by *I.* 31 the line joining the vertices of the triangles is parallel to the line in which the bases lie.

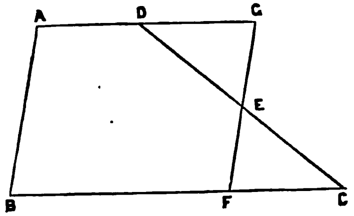
Ex. 4. The diagonals of a parallelogram divide it into four equal triangles.

Ex. 5. Apply Theor. 2, Cor. 3, to shew that the straight line joining the middle points of two sides of a triangle is parallel to the third side.

Ex. 6. Divide a triangle having its base angles acute into three parts, which shall together make up a rectangle having the same base but half the altitude of the triangle.

THEOR. 3. A trapezium is equal to a rectangle whose base is half the sum of the two parallel sides, and whose altitude is the perpendicular distance between them.

Let ABCD be a trapezium, having AD parallel to BC:



then shall the trapezium ABCD be equal to a rectangle whose base is half the sum of AD and BC, and whose altitude is the perpendicular distance between AD and BC.

Bisect CD at E, through E draw FEG parallel to BA, meeting BC and AD, produced if necessary, in F and G.

Then in the triangles DEG, CEF,
the angle GDE is equal to the alternate angle FCE, *I.* 23.
the angle DEG is equal to the angle CEF, *I.* 4.
and the side DE is equal to the side EC, *Constr.*
therefore the triangles are identically equal,
GD being equal to FC.

To each of the equal triangles FEC, DEG, add the figure ABFED,
then the trapezium ABCD is equal to the parallelogram ABFG.

Again, to each of the equals FC and DG add AD and BF,
then AD and BC are together equal to AG and BF;
but AG is equal to BF, *I.* 29.
therefore BF is half the sum of AD and BC.

But the parallelogram ABFG is equal to a rectangle whose base is BF and altitude the perpendicular distance between AD and BC, *II.* 1, *Cor.* 1.
therefore the trapezium ABCD is equal to a rectangle whose base

is half the sum of AD and BC, and whose altitude is the perpendicular distance between AD and BC.

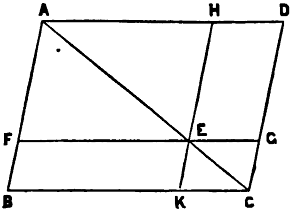
Q.E.D.

Ex. 7. If the sides AB, CD of the trapezium ABCD meet in H, prove that the triangles HBD, HAC are equal.

DEF. 3. *The straight lines drawn through any point in a diagonal of a parallelogram parallel to the sides divide it into four parallelograms, of which the two whose diagonals are upon the given diagonal are called* **parallelograms about that diagonal,** *and the other two are called the* **complements** *of the parallelograms about the diagonal.*

THEO. 4. The complements of the parallelograms about the diagonal of any parallelogram are equal to one another.

Let ABCD be a parallelogram, FK, HG the complements of the parallelograms AFEH, EKCG about the diagonal AC :



then shall the complement FK be equal to the complement HG.

Because a diagonal bisects a parallelogram *I.* 29.
therefore the triangles ABC, AFE, EKC are respectively equal to the triangles CDA, EHA, CGE ;

hence because the whole ABC is equal to the whole CDA, and the parts AFE and EKC are equal to the parts EHA, CGE, therefore the remainder FK is equal to the remainder HG.

Q.E.D.

Ex. 8. If E is not on the diagonal AC but within the triangle ABC, then FK will be less than HG.

Ex. 9. If in this case FG meets AC in L and HK meets it in M, the difference of the parallelograms FK, HG will be double of either of the triangles FMG or HLK.

DEF. 4. *All rectangles being identically equal which have two adjoining sides equal to two given straight lines, any such rectangle is spoken of as* **the rectangle contained by those lines.**

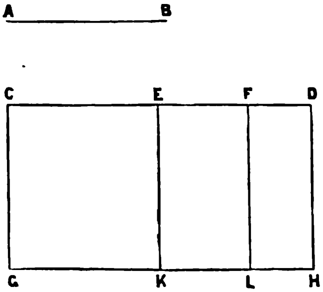
In like manner, any square whose side is equal to a given straight line is spoken of as **the square on that line.**

DEF. 5. *A point in a finite straight line is said to divide it* **internally**, *or, simply, to divide it: and, by analogy, a point in the line produced is said to divide it* **externally**; *and, in either case, the distances of the point from the extremities of the line are called the* **segments** *of the line.*

A straight line is equal to the sum or difference of its segments according as it is divided internally or externally.

THEOR. 5. The rectangle contained by two given lines is equal to the sum of the rectangles contained by one of them and the several parts into which the other is divided.

Let AB, CD be the two given lines, and let CD be divided into any number of parts CE, EF, FD:



then shall the rectangle contained by AB and CD be equal to the sum of the rectangles contained by AB and CE, AB and EF, AB and FD.

Draw CG at right angles to CD,
make CG equal to AB,
through G draw GH parallel to CD,
and through D, E, F draw DH, EK, FL parallel to CG.

The whole figure CH is equal to the sum of its parts CK, EL, FH.
Now CH is the rectangle contained by AB and CD,
since CG is equal to AB ; *Constr.*
CK is the rectangle contained by AB and CE,
since CG is equal to AB ;
EL is the rectangle contained by AB and EF,
since EK is equal to CG, and therefore to AB ; *I.* 29.
FH is the rectangle contained by AB and FD,
since FL is equal to CG, and therefore to AB.
Hence the rectangle contained by AB and CD is equal to the sum of the rectangles contained by AB and CE, AB and EF, AB and FD.

Q.E.D.

COR. 1. If a straight line is divided into two parts, the rectangle contained by the whole line and one of the parts is equal to the sum of the square on that part and the rectangle contained by the two parts.

This follows from Theor. 5 by taking the first line equal to one of two parts into which the second line is divided.

COR. 2. If a straight line is divided into two parts, the square on the whole line is equal to the sum of the rectangles contained by the whole line and each of the parts.

This follows from Theor. 5 by taking the two lines equal and dividing the second into two parts.

Ex. 10. Give algebraical identities corresponding to Theor. 5 and its Corollaries.

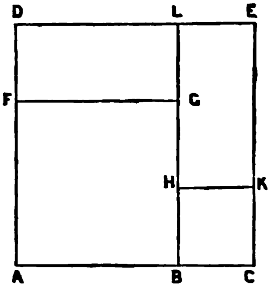
Ex. 11. Point out the relation between Theor. 5 and Multiplication in Arithmetic.

Ex. 12. If each of two straight lines is divided into any number of parts, the rectangle contained by the whole lines is equal to the sum of the rectangles contained by each part of the one taken together with each part of the other. Prove this (i.) by repeated applications of Theor. 5, (ii.) directly from Axiom *b*.

Ex. 13. Compare Ex. 12 with Multiplication in Algebra or Arithmetic.

THEOR. 6. The square on the sum of two lines is greater than the sum of the squares on those lines by twice the rectangle contained by them.

Let AB, BC, be the two lines, and let them be placed so that AC is their sum :



then shall the square on AC be greater than the sum of the squares on AB and BC by twice the rectangle contained by AB and BC.

Upon AC describe the square ADEC, and upon AB, BC describe the squares AFGB, BHKC.
Produce BG to meet DE at L.

Because right angles are equal,
therefore AF falls along AD, CK along CE,
and because right angles are supplementary,
therefore BH falls along BL.

Because AD is equal to AC, and AF to AB,
therefore the remainder FD is equal to the remainder BC;
in like manner KE is equal to AB.
Hence FL, being the rectangle contained by GF, FD, is also that contained by AB, BC;
and HE, being the rectangle contained by EK, KH, is also that contained by AB, BC;
so that FL and HE together are equal to twice the rectangle AB, BC.

Now the square AE exceeds the squares AG, BK together by the two rectangles FL, HE, therefore the square on AC is greater than the sum of the squares on AB, BC by twice the rectangle contained by AB and BC.

Q.E.D.

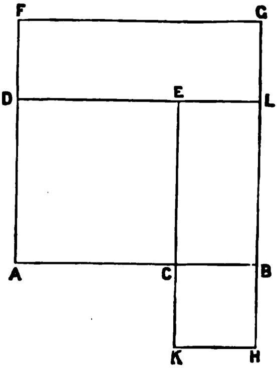
COR. The square on a line, which is divided internally into two segments, is greater than the sum of the squares on the segments by twice the rectangle contained by them.

Ex. 14. Give the algebraical formula corresponding to Theor. 6.

Ex. 15. If a straight line is divided into three parts, the square on the whole line is equal to the squares on the parts together with twice the rectangles contained by the parts taken two and two.

THEOR. 7. The square on the difference of two lines is less than the sum of the squares on those lines by twice the rectangle contained by them.

Let AB, BC be the two lines of which AB is the greater, and let them be placed so that AC is their difference:



then shall the square on AC be less than the sum of the squares on AB and BC by twice the rectangle contained by AB and BC.

Upon AC describe the square ADEC,
and upon AB, BC describe the squares AFGB, BHKC,
AG on the same side of AB as AE,
and BK on the opposite side;
produce DE to meet BG at L.

Because right angles are equal and supplementary,
therefore AF falls along AD, KC is on the same straight line with CE, as also HB with BG.

Because AF is equal to AB, and AD to AC,
therefore the remainder FD is equal to the remainder BC;
also because EC is equal to AC, and CK to CB;
therefore the whole EK is equal to the whole AB.
Hence FL, being the rectangle contained by FG, FD,
is also that contained by AB, BC;
and HE, being the rectangle contained by EK, KH,
is also that contained by AB, BC;
so that FL and HE together are equal to twice the rectangle AB, BC.

Now the square AE is less than the figure made up of the squares AG, BK by the two rectangles FL, HE,
therefore the square on AC is less than the squares on AB, BC by twice the rectangle contained by AB and BC.

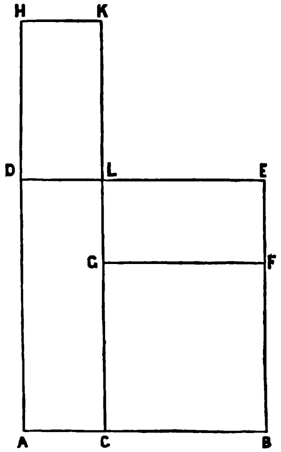
Q.E.D.

COR. The square on a line, which is divided externally into two segments, is less than the sum of the squares on the segments by twice the rectangle contained by them.

Ex. 16. Give the algebraical formula corresponding to Theor

THEOR. 8. The difference of the squares on two lines is equal to the rectangle contained by the sum and difference of the lines.

Let AB, BC be the two lines, BC the less lying upon AB he greater:



then shall the difference of the squares on AB and BC be equal to the rectangle contained by their sum and difference.

Upon AB, BC describe the squares ADEB, BFGC,
produce CG to meet DE at L,
and produce AD to H, making DH equal to BC;
complete the rectangle DHKL.

Then BF falls along BE, and CL, LK are in the same straight line.

Because BE is equal to AB, and BF to BC,
therefore EF is equal to AC,

and DL is equal to AC,
therefore EF is equal to DL;
also EL is equal to BC, and DH is equal to BC, *I.* 29.
therefore EL is equal to DH.
Hence DK, which is the rectangle contained by DL and DH, is equal to LF, which is the rectangle contained by EF and EL. *I.* 30, *Cor.*

Now the difference of the squares AE and CF is the figure made up of the rectangles AL, LF,
and LF has been shown to be equal to DK,
therefore the difference of the squares AE, CF is equal to the sum of the rectangles AL, DK, *i.e.* to the rectangle AK;
but AK is contained by AH, which is the sum of AB, BC, and AC, which is their difference;
therefore the difference of the squares on AB, BC is equal to the rectangle contained by their sum and difference.

Q.E.D.

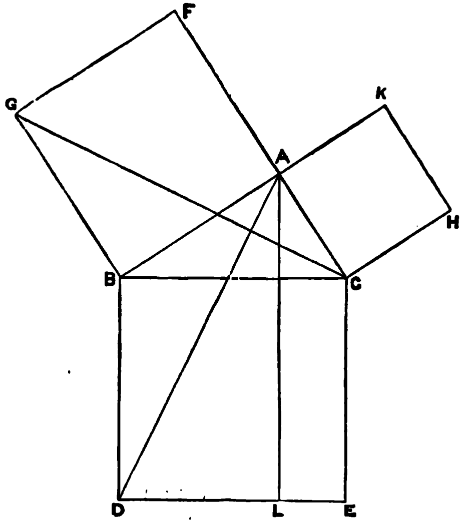
COR. If a straight line is divided either internally or externally at any point, the rectangle contained by the segments is equal to the difference of the squares on half the line and on the line between the point of division and the middle point of the line.

Ex. 17. Give the algebraical formula corresponding to Theor. 8.

Ex. 18. Prove that of all rectangles of given perimeter the square is the greatest.

THEOR. 9. In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the sides.

Let ABC be a triangle having the angle BAC a right angle:



then shall the square on BC be equal to the sum of the squares on AB and AC.

First Proof. Upon BC describe the square BDEC, upon AB the square BAFG, and upon AC the square ACHK;
draw AL parallel to BD, and join AD, CG.

The angle CBD is equal to the angle ABG, *I.* 1.
therefore the angle ABD is equal to the angle CBG. *Ax. d.*
Hence in the triangles ABD, GBC,
the side AB is equal to the side GB,
the side BD is equal to the side BC,
and the angle ABD is equal to the angle GBC,
therefore the triangles are identically equal. *I.* 5.

Now the angles BAC, BAF being right angles, FAC is a straight line; *I.* 3.

hence the square BF and the triangle GBC have the same altitude,
they are also on the same base GB,
therefore the square BF is double the triangle GBC. *II.* 2.
Similarly the rectangle BL is double the triangle ABD.

But it has been shown that the triangle ABD is equal to the triangle GBC,
therefore the rectangle BL is equal to the square BF,
which is the square on AB.

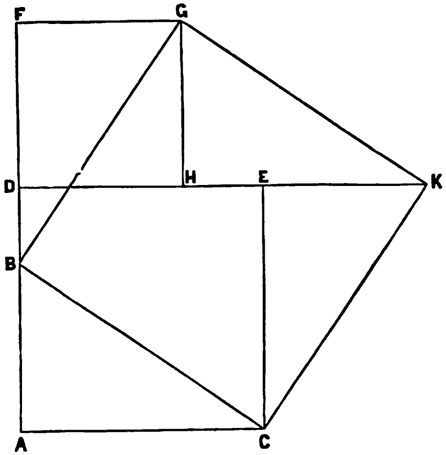
In like manner it may be shown that the rectangle CL is equal to the square on AC.

Now BL and CL make up the whole figure BE, which is the square on BC;
therefore the square on BC is equal to the sum of the squares on AB and AC.

Q.E.D.

Second Proof. On AC describe the square ADEC.
Produce AB to F making BF equal to AC or AD,
and therefore DF equal to AB. *Ax. d.*
On DF describe the square DFGH,
then this square is equal to the square on AB. *I.* 30, *Cor.*
Join BG, and produce DH to K, making HK equal to DE or AC.
Join CK, KG.

Then in the triangles CAB, BFG,
the side AC is equal to the side FB,
the side AB is equal to the side FG,
and the angle CAB is equal to the angle BFG,
therefore the triangles are identically equal. *I.* 5.
In like manner any pair of the triangles CAB, BFG, KHG, CEK may be proved to be identically equal.



Hence the figure CBGK is equilateral.

Again, the angle ECK being equal to the angle ACB, therefore the whole angle BCK is equal to the whole angle ACE, *Ax. d.*
therefore BCK is a right angle.
Hence the figure CBGK is the square on BC.

But because the triangle CEK is equal to the triangle BFG, and HGK to ABC,
therefore the square BK is equal to the figure made up of the squares AE and DG,
therefore the square on BC is equal to the sum of the squares on AB and AC.

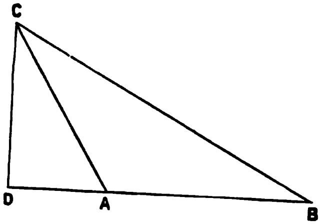
Q.E.D.

Ex. 19. Prove that in any right-angled triangle the rectangle contained by the hypotenuse and either part into which it is divided by the perpendicular from the right angle is equal to the square on the side adjacent to that part.

Ex. 20. Shew, by construction and reasoning similar to that in the First Proof, that in any triangle the rectangle contained by one side and the projection on it of another side is equal to that contained by the latter side and the projection on it of the former.

THEOR. 10. In an obtuse-angled triangle the square on the side opposite the obtuse angle is greater than the sum of the squares on the other two sides by twice the rectangle contained by either side and the projection on it of the other side.

Let ABC be a triangle having the angle BAC an obtuse angle, and let CD be perpendicular to BA produced:



then shall the square on BC be greater than the sum of the squares on BA and AC by twice the rectangle contained by BA and AD, the projection on BA of AC.

Because BD is the sum of BA and AD,
therefore the square on BD is greater than the sum of the squares on BA and AD by twice the rectangle BA, AD; *II.* 6.
to each add the square on DC,

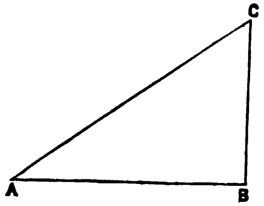
then the sum of the squares on BD and DC is greater than the sum of the squares on BA, AD and DC by twice the rectangle BA, AD;
but the square on BC is equal to the sum of the squares on BD and DC, *II.* 9.
and the square on AC is equal to the sum of the squares on AD and DC, *II.* 9.
therefore the square on BC is greater than the sum of the squares on BA and AC by twice the rectangle contained by BA and AD.
Q.E.D.

Ex. 21. Shew by making the angle BAC a straight angle that Theor. 6 is a limiting case of Theor. 10.

THEOR. 11. In any triangle the square on the side opposite an acute angle is less than the sum of the squares on the other two sides by twice the rectangle contained by either side and the projection on it of the other side.

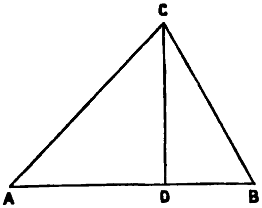
Let ABC be a triangle having the angle BAC an acute angle:
then shall the square on BC be less than the sum of the squares on BA and AC by twice the rectangle contained by BA and the projection on AB of AC.

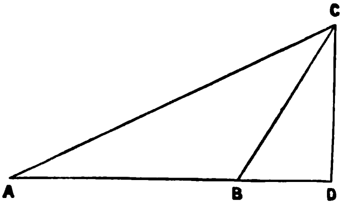
If the angle ABC is a right angle,



the rectangle contained by BA and the projection on AB of AC is the square on AB,
and the square on BC is less than the squares on BA and AC by twice the square on AB. *II.* 9.

But if the angle ABC is not a right angle,





let CD be perpendicular to AB or AB produced, so that AD is the projection on AB of AC.
Because BD is the difference of BA and AD,
therefore the square on BD is less than the sum of the squares on BA and AD by twice the rectangle BA, AD; *II.* 7.
to each add the square on DC,
then the sum of the squares on BD and DC is less than the sum of the squares on BA, AD and DC by twice the rectangle BA, AD;
but the square on BC is equal to the sum of the squares on BD and DC, *II.* 9.
and the square on AC is equal to the sum of the squares on AD and DC, *II.* 9.
therefore the square on BC is less than the sum of the squares on BA and AC by twice the rectangle contained by BA and AD.

Q.E.D.

COR. Conversely to Theorems 9, 10, 11, the angle opposite a side of a triangle is a right angle, an obtuse angle, or an acute angle, according as the square on that side is equal to, greater than, or less than the sum of the squares on the other two sides.

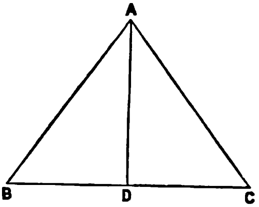
For in this group of Theorems the hypotheses are exhaustive and the conclusions are mutually exclusive. Hence the Rule of Conversion is applicable. *Introd.* § 9.

Ex. 22. Shew that Theor. 7 is a limiting case of Theor. 11.

THEOR. 12. The sum of the squares on two sides of a triangle is double the sum of the squares on half the base and on the line joining the vertex to the middle point of the base.

Let ABC be a triangle, D the middle point of the base BC: then shall the sum of the squares on AB and AC be double the sum of the squares on BD and DA.

If AD is at right angles to BC,

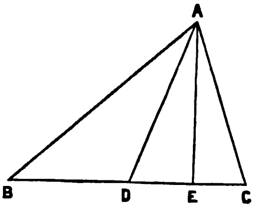


the square on AB is equal to the sum of the squares on BD and DA, *II.* 9.

and the square on AC is equal to the sum of the squares on CD and DA, *II.* 9.

that is, to the sum of the squares on BD and DA, since CD is equal to BD;
therefore the sum of the squares on AB and AC is double the sum of the squares on BD and DA.

But if AD is not at right angles to BC,



let ADB be an obtuse angle, and let DE be the projection on BC of AD.

Because ADB is an obtuse angle,
therefore the square on AB is greater than the sum of the squares on BD and DA by twice the rectangle BD, DE; *II.* 10.
and because ADC is an acute angle,
therefore the square on AC is less than the sum of the squares on CD and DA by twice the rectangle CD, DE, *II.* 11.
that is, the square on AC is less than the sum of the squares on BD and DA by twice the rectangle BD, DE,
since CD is equal to BD;
therefore the sum of the squares on AB and AC is double the sum of the squares on BD and DA.

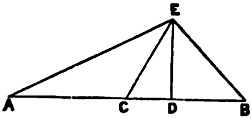
Q.E.D.

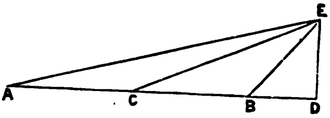
Ex. 23. The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

Ex. 24. The sum of the squares on the sides of any quadrilateral is greater than the sum of the squares on the diagonals by four times the square on the line joining the middle points of the diagonals.

THEOR. 13. If a straight line is divided internally or externally at any point, the sum of the squares on the segments is double the sum of the squares on half the line and on the line between the point of division and the middle point of the line.

Let the straight line AB be bisected at C and divided internally or externally at D:





then shall the sum of the squares on AD and DB be double the sum of the squares on AC and CD.

First Proof. Draw DE at right angles to AB, join EA, EB, EC.

The sum of the squares on AE and EB is double the sum of the squares on AC and CE; *II.* 12.
but the square on AE is equal to the sum of the squares on AD and DE, *II.* 9.
the square on EB is equal to the sum of the squares on BD and DE, *II.* 9.
and the square on CE is equal to the sum of the squares on CD and DE; *II.* 9.

therefore the sum of the squares on AD and DB together with twice the square on DE is equal to double the sum of the squares on AC, CD and DE,
therefore the sum of the squares on AD and DB is double the sum of the squares on AC and CD.

Q.E.D.

Second Proof. Because AD is the sum of AC and CD,
therefore the square on AD is greater than the sum of the squares on AC and CD by twice the rectangle AC, CD; *II.* 6.
and because BD is the difference of BC and CD,
that is, of AC and CD, since AC is equal to BC,
therefore the square on BD is less than the sum of the squares on AC and CD by twice the rectangle AC, CD; *II.* 7.
therefore the sum of the squares on AD and BD is double the sum of the squares on AC and CD.

Q.E.D.

COR. The sum of the squares on the sum and on the difference of two given lines is double the sum of the squares on those lines.

Ex. 25. The difference of the squares on the sum and difference of two given lines is equal to four times the rectangle contained by the lines.

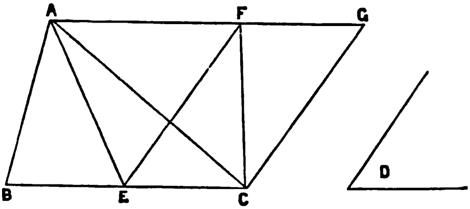
Ex. 26. Shew that Theor. 13 is a limiting case of Theor. 12.

SECTION II.

PROBLEMS.

PROB. 1. To construct a parallelogram equal to a given triangle and having one of its angles equal to a given angle.

Let ABC be the given triangle, and D the given angle: it is required to construct a parallelogram equal to ABC, and having an angle equal to D.



Bisect BC at E, *I. Prob.* 4.
at the point E in CE make the angle CEF equal to D, *I. Prob.* 5.
through A draw AFG parallel to BC, *I. Prob.* 6.
and through C draw CG parallel to EF; *I. Prob.* 6.
then shall FECG be the parallelogram required.

Join AE, FC.

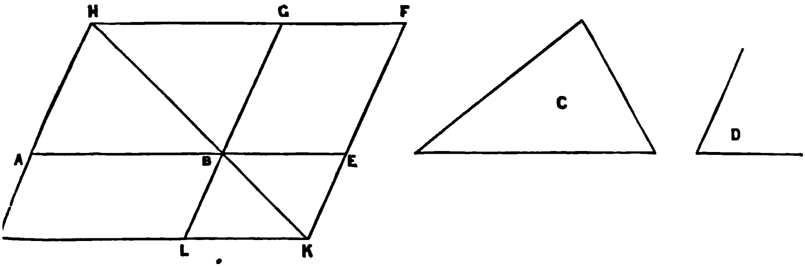
Then each of the triangles AEB, AEC, is equal to the triangle FEC, *II.* 2, *Cor.* 1.
therefore the triangle ABC is double the triangle FEC;
and because the diagonal FC of the parallelogram FECG bisects it, *I.* 29.
therefore the parallelogram FECG is also double the triangle FEC;

therefore the parallelogram FECG is equal to the triangle ABC, and it has the angle FEC equal to the given angle D.

Q.E.F.

PROB. 2. To construct a parallelogram on a given base equal to a given triangle and having one of its angles equal to a given angle.

Let AB be the given base, C the given triangle, and D the given angle:
it is required to construct on AB a parallelogram equal to C, and having an angle equal to D.



Construct a parallelogram GBEF equal to the triangle C, and having the angle GBE equal to D, *II. Prob.* 1.
so that BE may be in a straight line with AB.
Through A draw AH parallel to BG to meet FG produced in H. *I. Prob.* 6.
Join HB.
Because HA is parallel to FE, HB is not parallel to FE,
therefore HB and FE will meet if produced,
and on the side of BE,
since HB falls within the angle FHA.

Let them meet at the point K.
Through K draw KLM parallel to AE or HF, and meeting GB, HA produced in L and M; *I. Prob.* 6
then shall AL be the parallelogram required.

Because AL and GE are the complements of the parallelograms AG and LE about the diagonal HK of the parallelogram FM,
therefore AL is equal to GE, *II.* 4.
but GE was made equal to C,
therefore AL is equal to C.

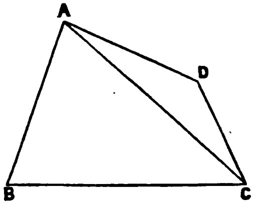
Again the angle ABL is equal to the vertically opposite angle GBE, *I.* 4.
and therefore is equal to D. *Constr.*

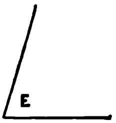
Hence AL is a parallelogram upon the base AB, equal to the triangle C, and having the angle ABL equal to D.

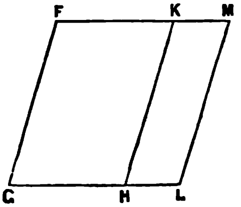
Q.E.F.

PROB. 3. To construct a parallelogram equal to a given rectilineal figure and having one of its angles equal to a given angle.

Let ABCD be a given figure of four sides, E the given angle:
it is required to construct a parallelogram equal to ABCD and having an angle equal to E.







Join AC. Construct the parallelogram FGHK equal to the triangle ABC, and having the angle FGH equal to E. *II. Prob.* 1. On KH construct the parallelogram KHLM equal to the triangle ACD, and having the angle KHL equal to E; *II. Prob.* 2. then shall the figure FGLM be the parallelogram required.

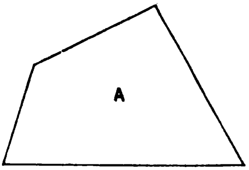
Because each of the angles FGH, KHL is equal to E, *Constr.*
therefore KHL is equal to FGH:
but FGH is supplementary to GHK, *I.* 28.
therefore KHL is supplementary to GHK,
therefore GH and HL are in the same straight line. *I.* 3.
Again, because FK, KM are each parallel to GHL,
and have a common point K,
therefore FKM is a straight line, *Ax.* 3.
hence GL and FM are parallel.
And FG and ML, being each parallel to KH, are parallel to one another, *I.* 24.
therefore the figure FGLM is a parallelogram.
Also FGLM is equal to ABCD,
since it is made up of FH, KL, which were made equal to the triangles ABC, ACD;
and the angle FGL was made equal to E;
therefore FGLM is the parallelogram required.

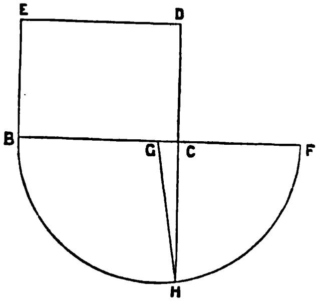
If the figure has five or more sides it can be divided into three or more triangles, and the same method of construction and proof may be used as for a four-sided figure.

Q.E.F.

PROB. 4. To construct a square equal to a given rectilineal figure.

Let A be the given rectilineal figure:
it is required to construct a square equal to A.





Construct the rectangle BCDE equal to the figure A. *II. Prob.* 3.

Then if BC is equal to CD, BD is the square required.
But if BC is not equal to CD,
produce BC to F making CF equal to CD;
bisect BF at G;
with centre G, and radius GB, draw a circle,
and produce DC to meet the circumference at H;
then shall the square on CH be equal to A.

Join GH.

Then the square on CH is equal to the difference of the squares on GH and GC. *II.* 9.
and GH is equal to GB,

therefore the square on CH is equal to the difference of the squares on BG and GC.

But the difference of the squares on BG and GC is equal to the rectangle contained by BC and CF, their sum and difference, *II.* 8.

i.e., to the rectangle BD, since CF is equal to CD.

Therefore the square on CH is equal to the rectangle BD,

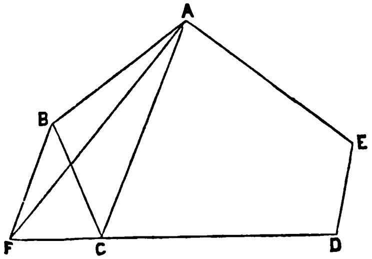
and therefore to the given figure A.

Q.E.F.

Ex. 27. By a construction similar to the above make a rectangle on a given base and equal to a given square.

PROB. 5. To construct a rectilineal figure equal to a given rectilineal figure and having the number of its sides one less than that of the given figure; and thence to construct a triangle equal to a given rectilineal figure.

Let ABCDE be a given rectilineal figure of five sides :
it is required to construct a four-sided figure equal to ABCDE.



Join AC,
through B draw BF parallel to AC, and let it meet DC produced at F ;

join AF;
then shall AFDE be equal to the figure ABCDE.
Because the triangles AFC and ABC are on the same base AC and of equal altitude,
therefore the triangle AFC is equal to the triangle ABC.
II. 2, *Cor.* 1.
To each add the figure ACDE,
then the figure AFDE is equal to the figure ABCDE.

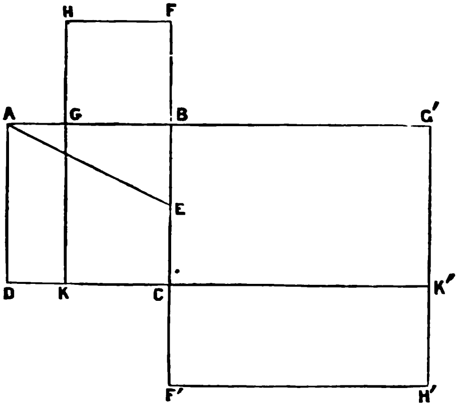
In like manner if EF be joined, and AG be drawn parallel to EF to meet DF produced at G, the triangle EGD will be equal to the figure AFDE, and therefore to the figure ABCDE.
Q.E.F.

Ex. 28. ABC is a triangle, D any point in BC produced, find a point E in AB such that the triangle EBD may be equal to ABC.

Ex. 29. Bisect a triangle by a straight line drawn through a given point in one of the sides.

PROB. 6. To divide a given straight line, either internally or externally, into two segments such that the rectangle contained by the given line and one of the segments may be equal to the square on the other segment.

Let AB be the given straight line:
it is required to divide it internally and externally so that the rectangle contained by AB and one segment may be equal to the square on the other segment.



On AB draw the square ABCD,
bisect BC at E,
join EA, and with centre E and radius EA draw a circle cutting BC produced in F and F′;
on BF and BF′ draw the squares BFHG and BF′H′G′:
then shall the rectangle contained by AB and AG be equal to the square on BG,
and the rectangle contained by AB and AG′ to the square on BG′.

Complete the rectangles FHKC and F′H′K′C.

Then the square on AB is equal to the difference of the squares on AE and BE; *II.* 9.
but the difference of the squares on AE and BE is equal to the rectangle contained by CF, their sum, and BF their difference, *i.e.*, to the figure FK, and, in like manner, to the figure F′K′; *II.* 8.
therefore the figure AC, which is the square on AB, is equal to each of the figures FK and F′K′.

Therefore the remainder AK, which is the rectangle contained by AB and AG, is equal to the remainder FG, which is the square on BG; and the whole AK′, which is the rectangle contained by AB and AG′, is equal to the whole F′G′ which is the square on BG′.

Q.E.F.

Ex. 30. Supposing AB to be a foot long, shew that the lengths of BG and BG′ in inches are the roots of the equation $x^2 + 12x = 144$, and hence calculate the lengths of BG and BG′ correctly to a tenth of an inch.

EXERCISES ON BOOK II.

31. Two equal triangles are on the same base and on opposite sides of it: shew that the base bisects the straight line joining their vertices.

32. The diagonals of a quadrilateral intersect at right angles: shew that the rectangle contained by the diagonals is double the quadrilateral.

33. The diagonals of a quadrilateral intersect at right angles: shew that the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other pair.

*34. ABC is an equilateral triangle and AD is perpendicular to BC: shew that the square on AD is equal to three times the square on BD.

35. Prove that the rectangle contained by the segments of a given straight line is greatest when the segments are equal.

36. If a straight line is divided into two parts, prove that the sum of the squares on the parts is least when the line is bisected.

37. A triangle whose base is one of the non-parallel sides of a trapezium, and vertex the middle point of the opposite side, is equal to half the trapezium.

38. D is the middle point of the base BC of the triangle ABC, BE bisects AD and meets AC at E: shew that the triangle BEC is double the triangle ABE.

*39. The difference of the squares on two sides of a triangle is equal to the difference of the squares on the segments of the base made by a perpendicular from the opposite vertex.

*40. If O be a point in the base BC, or BC produced, of the isosceles triangle ABC, the difference of the squares on OA and AB is equal to the rectangle contained by OB and OC.

41. If from the middle point of one of the sides of a right-angled triangle a perpendicular be drawn to the hypotenuse, the difference of the squares on the segments of the hypotenuse is equal to the square on the remaining side.

42. Prove that three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the lines drawn from the vertices to the middle points of the opposite sides.

43. The hypotenuses of three isosceles right-angled triangles form a right-angled triangle : shew that one of the isosceles triangles is equal to the sum of the other two.

44. In any quadrilateral the sum of the squares on the diagonals is double the sum of the squares on the lines joining the middle points of the opposite sides.

45. Prove, by constructions similar to that in the First Proof of Theor. 9, that in a triangle the sum of the squares on the sides containing an acute angle is greater, and on those containing an obtuse angle less, than the square on the other side.

46. Construct a square equal to the difference of two given squares.

47. Divide a given straight line into two segments such that the sum of their squares may be equal to the square on another given straight line. When must the straight line be divided externally?

48. Divide a given straight line into segments such that the difference of their squares may be equal to a given square.

49. Bisect a parallelogram by a straight line drawn through a given point.

50. Trisect a triangle by straight lines drawn from a given point in one of its sides.

51. Find a point in a given straight line such that the sum of the squares on its distances from two given points may be the least possible.

52. Find the locus of a point such that the sum of the squares on its distances from two given points may be equal to a given square.

53. Find the locus of a point such that the difference of the squares on its distances from two given points may be equal to a given square.

DEFINITIONS OF BOOK II.

DEF. 1. *The* **altitude** *of a parallelogram with reference to a given side as base is the perpendicular distance between the base and the opposite side.*

DEF. 2. *The* **altitude** *of a triangle with reference to a given side as base is the perpendicular distance between the base and the opposite vertex.*

DEF. 3. *The straight lines drawn through any point in a diagonal of a parallelogram parallel to the sides divide it into four parallelograms, of which the two whose diagonals are upon the given diagonal are called* **parallelograms about that diagonal,** *and the other two are called the* **complements** *of the parallelograms about the diagonal.*

DEF. 4. *All rectangles being identically equal which have two adjoining sides equal to two given straight lines, any such rectangle is spoken of as* **the rectangle contained by those lines.**

In like manner, any square whose side is equal to a given straight line is spoken of as **the square on that line.**

DEF. 5. *A point in a finite straight line is said to divide it* **internally,** *or, simply, to divide it; and by analogy, a point in the line produced is said to divide it* **externally**; *and, in either case, the distances of the point from the extremities of the line are called the* **segments** *of the line.*

Woodfall and Kinder, Printers, Milford Lane, Strand, London, W.C.